Estimation of Evoked Potentials

A Study of *a posteriori* "Wiener" Filtering and its Time-varying Generalization

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PROEFSCHRIFT

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Zonder de stimulans en daadwerkelijke steun van Servaas Notermans had een medisch-fysische werkgroep binnen de klinische neurofysiologie niet van de grond kunnen komen en was dit onderzoek ondenkbaar geweest. Daarnaast heb ik altijd een beroep kunnen doen op mijn oude thuisbasis: het laboratorium voor medische fysica en biofysica van de Nijmeegse Universiteit. Ik denk daarbij vooral aan de constructieve discussies met Gerard Uyen.

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Het proefschrift en de daaraan voorafgaande manuscripten werden met grote zorg uitgewerkt en getypt door Janny Kuppens-Dirksen. Ook Justine Renders-van den Broeke leverde vele bijdragen. Het merendeel van de illustraties werd verzorgd door Nico Dijkstra; de fotografie was in handen van Frans Dehue, beiden van de foto-, film- en tekenafdeling van het Instituut voor Neurologie. Voor problematische fotografische reproducties klopten wij nimmer tevergeefs aan bij de afdeling fotografie van de A-faculteiten (hoofd: de heer S.G.Spaan). In de eindfase van dit proefschrift werd een deel van het fotografische werk verzorgd door de centrale medische fotografie (hoofd: de heer A.T.A. Reynen). Dennis Cotton droeg zorg voor de engelse correctie van de publicaties en de overige tekst in dit proefschrift.

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Aan Jola, Sabina, Liza
en mijn ouders

Contents

Chapter 1

Introductory Notes on Evoked Potentials and Summary of this Study

1. Some Historical Aspects

Evoked potentials may be defined as the electrical voltage fluctuations which can be recorded from parts of the nervous system in response to stimulation of sensory modalities. One may distinguish between evoked potentials from the *peripheral* and the *central* nervous system. For the latter type a further subdivision can be made into *spinal*, *brainstem*, and *cortical* evoked potentials, according to the (assumed) structures from which the responses derive.

Although the study of human evoked potentials began shortly after Berger recorded the first human electroencephalogram (Berger, 1929), it was not until 1947 that Dawson could unequivocally demonstrate that specific cortical evoked potentials could be recorded from the scalp, following (somato-)sensory stimulation (Dawson, 1947). The explanation for this delayed development lies in the fact that cortical evoked potentials are usually rather small in amplitude as compared to the ongoing spontaneous EEG and cannot be visualized unless some form of signal processing is employed.

Dawson's first evoked potential recording was made possible by means of the photographic *superimposition* technique, by which evoked potentials were enhanced through the superimposition of successive responses on a cathode ray oscilloscope. In subsequent years Dawson introduced the principles of signal averaging into neurophysiology and described an ingenious electro-mechanical machine for performing this operation automatically (Dawson, 1951; 1954). In the latter paper the capabilities of the method for extracting cortical sensory evoked potentials from an overwhelming background activity were clearly demonstrated. This major development continued with a contribution from Barlow (1957), describing an electronic averager, based on an analog cross-correlator system, followed by papers from Clark et al. (1961), describing a digitally operated device (the *A*verage *R*esponse *C*omputer), and Clark and Molnar (1964), who presented the *L*aboratory *In*strument *C*omputer, which can be regarded as the forerunner of the laboratory computers of today.

A breakthrough to the more widespread application of evoked potentials was formed by the *commercial availability* of the *C*omputer of *A*verage *T*ransients and similar equipment produced in the early sixties. These technological advances initiated many studies on the characteristics, the origin and the usefulness of *cortical* evoked potentials in the fields of neurophysiology, clinical medicine, psychophysics and psychology.

A further significant progress in the study of *visual* evoked potentials was the introduction of *patterned* light stimuli (Spehlmann, 1965; Spekreyse, 1966; Rietveld et al., 1967; Spekreyse et al., 1973). Unlike evoked potentials elicited by diffuse flashes, these "pattern evoked potentials" proved to be reliable both in form and latency between subjects (Harter and White, 1968) and to be useful in clinical practice (Halliday et al., 1972; Desmedt, 1977b).

Subsequently, another landmark in the history of evoked potentials was the discovery that short-latency *brainstem* components of the *auditory* evoked potential could be recorded from the human scalp (Sohmer and Feinmesser, 1967; Jewett et al., 1970; Jewett and Williston, 1971). The past decade has shown great progress in the understanding of the origin and the clinical usefulness of these early potentials.

Similar developments for *somatosensory* evoked potentials demonstrated the possibilities and usefulness of recording *spinal* and early *brainstem* components (Liberson and Kim, 1963; Cracco, 1972; 1973; Desmedt, 1980). More recently, short-latency *visual* evoked potentials have been described as well (van Hasselt, 1972; Cracco and Cracco, 1978). As yet, their precise origin remains uncertain and their clinical significance is still to be established.

2. Clinical Applications

At the present time the evoked potentials commonly employed in diagnostic procedures are those of the auditory, somatosensory and visual systems. These systems are relatively easy to stimulate, e.g. by means of an acoustic click, a brief electric shock or a flashing light pattern. In contrast, stimulation of the olfactory and gustatory systems has proven to be technically difficult and research in these areas is still in an early stage (see e.g. Plattig and Kobal, 1979).

During the past decade, the measurement and analysis of evoked potentials has reached a well established position within the arsenal of noninvasive electro-diagnostic procedures. Numerous papers in almost every field of clinical medicine, notably in Clinical Neurophysiology, Neurology and Neurosurgery, Audiology, Ophthalmology, Orthopedics, Pediatrics and Psychiatry, have described the clinical usefulness of evoked potentials in evaluating peripheral and central nervous system functioning, as well as auditory and visual functioning. Some of these applications have been illustratively listed in Table 1.1.

Diagnosis and *differentiation* of disorders
of the peripheral and central nervous system
- demyelinating diseases
- tumor localization in spinal cord and intracranial pathways
- inflammatory and traumatic lesions
- neuropathies

Evaluation and prognosis of mental function
- in congenital disorders
- in degenerative disorders

Estimation of auditory function
- particularly in newborns and infants
- objective audiometry

Estimation of visual function
- particularly in infants
- objective optometry

Intra-operative *monitoring*
- during neurosurgical procedures
- during orthopedic procedures
- during open-heart surgery
- for controlling anaesthetic depth

Intensive Care *monitoring*
- post-operative
- in coma
- in assessing cerebral death

Table 1.1 Some typical applications of Evoked Potentials

For a more detailed picture, the interested reader may consult one of the many books on evoked potential applications that have appeared recently, e.g. Desmedt, 1977 a,b; Lehmann and Callaway, 1979; Barber, 1980; and Desmedt, 1980.

Obviously the evoked potential field has demonstrated a rapidly increasing number of applications and it is likely that the advent of microprocessor technology, with its promises of low cost instrumental innovations, will further stimulate their widespread use.

3. Current Processing Methodology

As stated already in section 1, the main problem in evoked potential measurements lies in the fact that these potentials are in general very small as compared to the background activity. Whereas the amplitude of cortical evoked potentials is normally in the order of 10-30 μV ("peak-to-peak"), and the peak-to-peak amplitude of brainstem and spinal evoked potentials rarely exceeds a few microvolts, the spontaneous activity usually reaches amplitudes from 50 to 100 microvolts.

The most commonly employed solution to the above measurement problem is the use of signal (or "ensemble") *averaging*. In this method an identical stimulus is applied repeatedly and the evoked potential is obtained by averaging over the collection *(ensemble)* of individual responses. The assumptions underlying this technique are that (1) the evoked potential waveform (termed the "signal") is additive to the background activity (termed the "noise"); (2) the signal remains identical and occurs with the same latency from trial to trial and (3) the noise varies randomly and is unrelated to the stimulus. Under the conditions of stationary, uncorrelated, noise it can be proved mathematically that the "signal-to-noise ratio" (SNR) improves by the square root of the number of averaged responses.

The method of signal averaging is usually employed for the measurement of *transient* evoked potentials. Implicit in the measurement of such potentials is the assumption that the stimuli are sufficiently spaced apart in time, such that the generating system (e.g. the brain) returns to its "resting state" between stimuli.

Another approach in processing cortical evoked potentials involves the use of periodic stimulation at relatively high frequencies, such that the brain does, intentionally, not return to its "resting state". Evoked potentials thus elicited are called *steady-state* evoked potentials. They may be measured and analyzed by using frequency domain techniques (see e.g. Spekreyse et al., 1977 and Regan, 1977). However, these methods have found little application in clinical practice and will not be discussed further.

In view of the small signal-to-noise ratio of evoked potentials, the averaging technique often requires a large number of stimuli to be presented. For example, the measurement of cortical evoked potentials usually requires in the order of a hundred stimuli, while measurement of brainstem and spinal evoked potentials requires about a thousand or more stimuli. These large numbers may give rise to both practical and fundamental measurement problems.

Firstly, there is sometimes an imperative need for short recording sessions. This occurs, e.g., in investigating young children, whose generally limited reserve of patience restricts the possible duration of the investigation. Another example is the use of evoked potentials for intra-operative monitoring during (neuro-) surgical procedures. In this case one should ideally like to have almost instantaneous results.

Secondly, and more fundamentally, the assumptions previously made with respect to the "signal" and the "noise" become more vulnerable as the duration of the recording session increases. The evoked potentials of cortical origin, in particular, show variations which are related to factors that are difficult to control in the long run, such as attention and habituation. Likewise, the stationarity of the background EEG becomes a problem over long periods of time. Therefore these factors also set bounds to the duration of the investigation and thus to the number of admissible stimuli. Although, as a consequence, the signal-to-noise ratio may show insufficient improvement, it is no uncommon finding in practice that continued averaging over long periods of time *degrades*, rather than further enhances the averaged evoked potential waveform.

4. Advanced Processing Methodology

The measurement problem as discussed in the previous section has induced two main lines of research aimed at improving the estimation of averaged evoked potentials (Table 1.2). The *first line* is concerned with the question as to what extent the basic assumptions underlying the averaging technique are valid in practice. Several alternative estimation methods have been suggested which have in common that prior to ensemble averaging some form of pre-processing on individual responses is performed. The type of pre-processing depends on the evoked potential model actually assumed.

Cross-correlation averaging (Woody, 1967; Wastell, 1977) is based on the assumption that the evoked potential waveform is invariant and additive to the background noise, but has imperfect time-lock to the stimulus, i.e. a certain time-*jitter* occurs. Assuming this model, improved estimates of the evoked potential can be obtained by aligning individual responses prior to averaging. This alignment is carried out by means of an iterative cross-correlation procedure.

Latency corrected averaging (McGillem and Aunon, 1977; Aunon and Sencaj, 1978) assumes a more complicated evoked potential model, in that individual peaks and troughs of the evoked potential waveform are thought to vary independently in both amplitude and latency. The authors suggest that an improved estimate of the evoked potential can be obtained by first applying to the individual responses an adaptive filter, designed to minimize the background noise, followed by a statistical procedure for grouping peaks and troughs to-

Cross-correlation averaging	(Woody, 1967)	techniques for optimizing ensemble averaging
Selective averaging	(Pfurtscheller and Cooper, 1975)	
Latency-corrected averaging	(McGillem and Aunon, 1977)	
"Wiener" filtering	(Walter, 1969)	*a posteriori* least mean square techniques
Time-varying filtering	(de Weerd, 1977)	

Table 1.2 Signal analysis techniques in average evoked potential studies

gether, after which these are aligned and averaged. This procedure leads to somewhat unnatural evoked potential waveforms broken up into separate peaks and troughs.

Selective averaging [1)] (Pfurtscheller and Cooper, 1975) is based on a similar evoked potential model to the one used by McGillem and Aunon (1977). In this case, however, attention is focussed on only a selected component of the entire evoked potential waveform. Cross-correlation techniques are again applied to estimate the amplitude and latency of such a component in individual responses. As a next step, only those responses that meet certain criteria with respect to latency and amplitude are averaged.

The *second line* of research deals with the (rather fundamental) question of whether better estimators exist than ensemble averaging [2)], accepting all assumptions underlying the latter technique to be valid. Two related estimation methods have been suggested which have in common that *posterior* to ensemble averaging some form of adaptive filtering of the averaged waveform is performed.

A posteriori "Wiener" filtering (Walter, 1969; Doyle, 1975) estimates from the ensemble of individual responses the underlying power density spectra of the "signal" and the "noise". These power spectra are then used to construct, according to the Wiener Filter theory (Wiener, 1949), an "optimal" time-*invari-*

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1) This method is different from that proposed, with the same name, by Ungan and Basar (1976). Their method relies on a visual inspection of all individual responses, which is an impracticable procedure in a clinical setting; it has therefore been omitted from Table 1.2.

2) This question is posed here in a rather loose sense. A more precise formulation is given in later chapters.

ant filter, which further reduces the remaining noise in the averaged evoked potential waveform. This is accomplished by weighing the spectral components of the averaged waveform according to the signal-to-noise ratio at individual frequencies.

Time-varying filtering (de Weerd et al., 1977) proceeds along similar lines taking, however, the transient structure of evoked potential waveforms into account. Specifically, this implies that the averaged evoked potential waveform is processed by a time-*varying* filter which is based on estimated time-varying spectra of the signal and the noise.

The development of this second line of research and the introduction and elaboration of the method of time-varying filtering will form the central themes of this thesis.

5. Summary of the Present Investigation

Following the earlier work of Walter (1969) and Doyle (1975) on *a posteriori* "Wiener" filtering *(a.p.w.f.)*, the present study aims at investigating adaptive filtering methods that provide (in the mean-square error sense) an improved estimation beyond ensemble averaging, *making no other assumptions regarding signal and noise than those already underlying the averaging technique* (see section 3).

In an early stage of the investigation it became obvious that time-*invariant* filtering methods, such as a.p.w.f., are not well-suited for improving the dynamic, transient, waveform of averaged evoked potentials (de Weerd et al., 1977). However, before elaborating a time-*varying* generalization of the a.p.w.f. method, it seemed appropriate to investigate some controversial aspects of that method which had been reported in the literature (e.g. Albrecht and Radil-Weiss, 1976; Hartwell and Erwin, 1976; Ungan and Basar, 1976, and Strackee and Cerri, 1977). Therefore, a theoretical and practical analysis was made of a.p.w.f. (Chapter 2) and of methods for estimating the power spectra of signal and noise from ensembles where the signal appears as an identically repeating waveform in the presence of stationary, additive noise (Chapter 3). The theoretical insights resulting from these studies, combined with practical experience of the a.p.w.f. method resulted, in a later stage, in a tutorially oriented paper explaining some of the pitfalls of the method and providing qualitative arguments for using time-varying, rather than time-invariant, filtering methods in estimating transient evoked potentials (Chapter 4).

Meanwhile, a methodological study had been undertaken to establish appropriate means for obtaining time-varying spectra of transient evoked potential waveforms (Chapter 5). This study formed the basis for the *time-varying filtering (t.v.f.)* method, which is introduced in Chapter 6 and elaborated in detail in Chapter 7. In addition, Chapter 6 reviews some of the basic problems encountered in evoked potential estimation and it explains, on a theoretical basis, why ensemble averaging is, in general, *not* the best estimator in the mean square error sense.

The method of time-varying filtering, as it is reported in this study, has been applied in clinical practice now for several years. It has proved its usefulness in evaluating human evoked potentials of the auditory, somatosensory and visual systems. Some illustrative results of these applications are described in Chapter 8.

Although the organization of chapters in this book thus represents logical steps towards more refined and clinically useful estimation methods, the actual sequence is not an adequate one for readers unfamiliar with the present material. It may be recommended to start with Chapters 4 and 6, if a more general discussion on the background and merits of a.p.w.f. and t.v.f. and their application to the estimation of transient evoked potentials is desired.

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Chapter 2

Theory and Practice of *a posteriori* "Wiener" Filtering of Average Evoked Potentials

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Abstract

In this study a theoretical and practical analysis of the technique of *a posteriori* "Wiener" filtering of averaged evoked potentials is presented. It is shown that spectral smoothing plays a central role in obtaining a reliable estimate. Attention is paid to some practical problems that arise when the method is applied in discrete time. In an illustrative case study results are compared to theoretical Wiener filtering, while the influence of slow amplitude modulation has also been investigated. Some of the conflicting results regarding the method that have been reported in several recent papers are discussed. It is concluded, that when the method is applied with prudence, it may lead to an improved estimation of those evoked potentials that have an (almost) periodic character. If the latter assumption is not valid, application of a time-varying filtering technique may be recommended instead.

1. Introduction

Several recent reports have dealt with the application of *a posteriori* "Wiener" filtering (*a.p.w.f.*) to averaged evoked potentials in order to obtain a further improvement in the signal-to-noise ratio after ensemble averaging (Nogawa et al, 1973; Basar et al, 1975; Ungan and Basar, 1976; Hartwell and Erwin, 1976; Albrecht et al, 1977). The idea of applying the Wiener theory to averaged evoked potentials was first reported by Walter (1969), who formulated an *a posteriori* filter, which some years later was modified by Doyle (1975). Obviously there is no general agreement with respect to the usefulness of a.p.w.f. since several contradictory papers regarding the application of this method have appeared, e.g. Hartwell and Erwin (1976) as against Ungan and Basar (1976). It seems that the main reasons for the inconsistent findings can be attributed to three major problems:

a) Invalid assumptions concerning the nature of the signal. It is well known, that one of the assumptions on which Wiener filtering is based, concerns the *random* and stationary character of *both* the signal and the noise. However, in the usual evoked potential model [see (2.1)], the signal is considered *deterministic*. McGillem and Aunon (1977), applying a time domain technique essentially equivalent to Wiener filtering, concluded that the formalism for deterministic signals does not deviate significantly from the one used for random stationary signals. But the fact that Wiener filtering is a time-*in*dependent filtering implies that it is not well suited for the optimal filtering of *transient* evoked potentials. However, once an appropriate theoretical basis for a.p.w.f. is established, the method may be generalized to the technique of time-varying filtering, by which transient waveforms can be handled (de Weerd, 1981; de Weerd and Kap, 1981).

b) Invalid assumptions concerning the behaviour of signal and noise. A further assumption on which Wiener filtering is based is that signal and noise are additive. In addition, the a.p.w.f. formulation assumes that signal and noise are uncorrelated. Ungan and Basar (1976) have summarized some observations from which it can be concluded that these suppositions are not always valid in evoked potential analysis. Sayers and Beagley (1974) come to a similar conclusion in their study of the auditory evoked potential, and they suggest that an effective stimulus possibly acts mainly to phase-control the spontaneous activity (i.e. the noise). In reply to a paper of von Specht and Kevanishvili (1976), they state more precisely that the supposition of additivity of evoked potential and noise is nevertheless a useful model, which, however, becomes increasingly less useful as the stimulus threshold is approached. As already pointed out by Ungan and Basar (1976), the necessity of the foregoing assumptions applies equally well to the technique of ensemble averaging and therefore their validity should already be judged at that stage of the signal processing.

c) Blind use of the "Wiener" filtering theory. Intentionally Wiener is put between quotation marks here, since it should be realized that Wiener's filter theory is based on the *knowledge* of the spectra of signal and noise, rather than on their *estimates*. Most recently Strackee and Cerri (1977) have reported a theoretical study concerning the consequences of the latter problem, resulting in a valuable mathematical formulation of the statistical problems associated with estimated a.p.w.f. The authors conclude that a.p.w.f., as applied to the *averaged* time signal, does not give any improvement at all, while application to individual responses does improve the estimate. However, it will be shown that it is possible indeed to improve the averaged time signal as well, in particular when spectral smoothing techniques are used.

The aim of the present study is to present a theoretical and practical evaluation of the a.p.w.f. method, with and without smoothing, for deterministic signals, perturbed by additive, uncorrelated noise. Special attention is paid to the statistical problems related to the estimation of the *a posteriori* filter transfer function. At the same time this study is intended to form the basis for a generalization of the *a posteriori* "Wiener" filter theory, i.e. the technique of *time-varying filtering* (de Weerd et al., 1977; de Weerd, 1981; de Weerd and Kap, 1981). After a summary of the a.p.w.f. technique (section 2) the statistical behaviour of the estimated filter is analysed (section 3), and the necessity of spectral smoothing is demonstrated (section 4). Section 5 deals with the problems of aliasing, leakage and periodic convolution, which result from the application of a.p.w.f. in discrete time. In a case study (section 6) a simulated signal perturbed by additive white noise is used to analyse the performance of the a.p.w.f. as compared to averaging and theoretical Wiener filtering. In this section attention is also paid to the effects of signal distortion, caused by application of the method, and to the influence of slow signal amplitude variations. Finally, in the last section, the results of the present study are summarized and the applicability of the method is discussed in general.

2. Summary of the *a posteriori* "Wiener" Filtering Technique

Let

$$x_i(t) = s(t) + n_i(t) \qquad i = 1,2, \ldots..N \tag{2.1}$$

where $s(t)$ is the evoked potential waveform, $n_i(t)$ a stationary noise process with zero mean and N the number of ensemble elements.

As a further simplification we will assume that $s(t)$ and $n_i(t)$ are uncorrelated. Ensemble averaging of $x_i(t)$ leads to

$$\bar{x}(t) = s(t) + \frac{1}{N}\sum_{i=1}^{N} n_i(t) \qquad 0 \leq t \leq T \tag{2.2}$$

where T is the duration of the evoked potential, and the dash above the x represents ensemble averaging of N elements, i.e. $\frac{1}{N}\sum_{i=1}^{N} x_i$. This notation will be used throughout the paper.

Let the Fourier Transform of $\bar{x}(t)$ be denoted by $\bar{X}(\omega)$ and the power density spectrum of $\bar{x}(t)$ by $\Phi_{\overline{xx}}(\omega) = \bar{X}^*(\omega)\bar{X}(\omega)$. Then, assuming noise at different stimulus presentations to be uncorrelated,

$$E\{\Phi_{\overline{xx}}(\omega)\} = \Gamma_{\overline{xx}}(\omega) = \Gamma_{ss}(\omega) + \frac{1}{N}\Gamma_{nn}(\omega) \tag{2.3}$$

where, in general, Γ represents the expectation of the power density spectrum Φ, so that $\Gamma_{\overline{xx}}$, Γ_{ss} and Γ_{nn} represent the theoretical power density spectra of the average, the signal and the noise respectively.

By averaging the power density spectra of all the individual ensemble elements, we obtain

$$E\{\overline{\Phi_{xx}(\omega)}\} = \Gamma_{xx}(\omega) = \Gamma_{ss}(\omega) + \Gamma_{nn}(\omega) \tag{2.4}$$

where Γ_{xx} is the theoretical power density spectrum of x(t).

From (2.3) and (2.4) the spectra of signal and noise can be expressed in terms of $\Gamma_{\overline{xx}}$ and Γ_{xx}:

$$\Gamma_{ss}(\omega) = \frac{N}{N-1}\{\Gamma_{\overline{xx}}(\omega) - \frac{1}{N}\Gamma_{xx}(\omega)\} \tag{2.5}$$

and

$$\Gamma_{nn}(\omega) = \frac{N}{N-1}\{\Gamma_{xx}(\omega) - \Gamma_{\overline{xx}}(\omega)\} \tag{2.6}$$

The transfer function of the theoretical *a posteriori* Wiener filter is given by (Doyle, 1975):

$$H_N(\omega) = \frac{\Gamma_{ss}(\omega)}{\Gamma_{ss}(\omega) + \frac{1}{N}\Gamma_{nn}(\omega)} \tag{2.7}$$

which, by substitution of (2.3) and (2.5) can be rewritten as

$$H_N(\omega) = \frac{N}{N-1}\left\{1 - \frac{1}{N}\frac{\Gamma_{xx}(\omega)}{\Gamma_{\overline{xx}}(\omega)}\right\} \tag{2.8}$$

Usually the techniques of ensemble averaging and a.p.w.f. are compared on the basis of the mean square error (MSE):

$$\varepsilon = \frac{1}{T}\int_0^T \{\hat{s}(t) - s(t)\}^2\, dt \tag{2.9}$$

where $\hat{s}(t) = \bar{x}(t)$ in case of averaging and $\hat{s}(t)$ = a filtered form of $\bar{x}(t)$ in case of a.p.w.f.

With (2.2) and Parseval's theorem, the expected value of the MSE after ensemble averaging $\varepsilon_A(N)$ follows from

$$E\{\varepsilon_A(N)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{N} \Gamma_{nn}(\omega) d\omega \tag{2.10}$$

If the spectra of signal and noise were known exactly, the expected value of the MSE after (theoretical) a.p.w.f. $\varepsilon_T(N)$ should equal (cf. Deutsch, 1965):

$$E\{\varepsilon_T(N)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\frac{1}{N} \Gamma_{nn}(\omega) \cdot \Gamma_{ss}(\omega)}{\Gamma_{ss}(\omega) + \frac{1}{N} \Gamma_{nn}(\omega)} d\omega \tag{2.11}$$

By defining

$$\Theta(\omega) = \frac{\Gamma_{ss}(\omega)}{\Gamma_{nn}(\omega)} \tag{2.12}$$

which we will call the *signal-to-noise power density ratio* (SNPDR), (2.11) may be rewritten as

$$E\{\varepsilon_T(N)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{N} \Gamma_{nn}(\omega) \cdot \left\{ \frac{1}{1 + \frac{1}{N\Theta(\omega)}} \right\} d\omega \tag{2.13}$$

Evidently (2.13) is a function of $N\Theta(\omega)$, representing the SNPDR *after averaging* [1].

Comparing the methods of ensemble averaging and a.p.w.f., it follows from (2.10) and (2.13) that the ratio

$$R = \frac{E\{\varepsilon_A(N)\}}{E\{\varepsilon_T(N)\}} = \frac{\int_{-\infty}^{\infty} \Gamma_{nn}(\omega) d\omega}{\int_{-\infty}^{\infty} \Gamma_{nn}(\omega) \left\{ \frac{1}{1 + \frac{1}{N\Theta(\omega)}} \right\} d\omega} \tag{2.14}$$

becomes progressively larger with decreasing $N\Theta(\omega)$. The ratio defined by (2.14) is very similar to the "estimation error ratio" introduced by Doyle (1977). The fact, that this ratio increases with decreasing $N\Theta(\omega)$ does *not* obviously imply that a.p.w.f. also becomes progressively more effective, as stated by Doyle (1977) in the same paper. This point may be clarified as follows. In general, ensemble averaging does not affect the signal. This also

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[1] This follows from $N\Theta(\omega) = \Gamma_{ss}(\omega) / \frac{1}{N} \Gamma_{nn}(\omega)$

follows from (2.10), which shows that the error is independent of the signal spectrum. On the other hand, it follows from (2.12) and (2.7) that the a.p.w.f. transfer functions equals

$$H_N(\omega) = \frac{1}{1 + \frac{1}{N\Theta(\omega)}} \tag{2.15}$$

which implies that the reduction in the estimation error results from a selective suppression, depending on the SNPDR, of *both* signal and noise. From (2.11) it follows, that when the spectra of signal and noise do overlap, the application of a.p.w.f. causes distortion in the signal which becomes progressively larger with decreasing $NO(\omega)$. In view of this, one should be careful in comparing the methods of averaging and a.p.w.f. solely on the basis of their respective mean-square errors, since both methods operate on the data in quite different ways. This is also shown in an illustrative example in Section 6.4.

From the foregoing, we can intuitively indicate a range for $N\Theta(\omega)$ where application of a.p.w.f. may be useful. For low $N\Theta(\omega)$, this range is bounded by an inadmissible distortion that may emerge, while for high $N\Theta(\omega)$ the method becomes ineffective, as appears from (2.14) and (2.15). Therefore, we will focus our attention primarily on the range $0.1 \leq N\Theta(\omega) \leq 10$, with corresponding values $0.09 \leq H_N(\omega) \leq 0.9$, rather than on the range of $N\Theta(\omega) > 10$ to which the conclusions of Strackee and Cerri (1977) have been restricted.

Thusfar, we have assumed that the spectra of signal and noise were known *a priori*. In general, as a matter of fact, this knowledge is lacking, and the filter transfer function must be estimated using the spectra $\overline{\Phi_{xx}}(\omega)$ and $\Phi_{\bar{x}\bar{x}}(\omega)$ instead of the theoretical spectra $\Gamma_{xx}(\omega)$ and $\Gamma_{\bar{x}\bar{x}}(\omega)$.
It will be shown that the crucial point is that (2.8) is replaced by

$$\hat{H}_N(\omega) = \frac{N}{N-1}\left\{1 - \frac{1}{N}\,\frac{\overline{\Phi_{xx}}(\omega)}{\Phi_{\bar{x}\bar{x}}(\omega)}\right\} \tag{2.16}$$

i.e. that the *expected* values in (2.8) are replaced by the *estimated* values based on an ensemble size N. It should be further noted that substitution of the *estimated* spectra in (2.11) instead of the *theoretical* spectra leads to an incorrect value for the estimated mean-square error, unlike what has been suggested by Albrecht and Radil-Weiss (1976) and Albrecht et al (1977). The correct determination of this error follows from (2.9), whereby

$$\hat{s}(t) = F^{-1}\,\{\hat{H}_N(\omega)\;.\;\bar{X}(\omega)\} \tag{2.17}$$

where $\hat{s}(t)$ represents the *a posteriori* "Wiener" filtered estimate of the signal s(t) and F^{-1} denotes the inverse Fourier transform. Once more, Wiener is put between quotation marks, since estimation according to (2.17) is, in general, *not optimal* in the mean-square error sense. This is due to the fact that $H_N(\omega)$ is an estimate itself, as is shown in the next section.

3. Analysis of the Estimated *a posteriori* Filter

From (2.16) it follows, that for a statistical analysis of the estimated *a posteriori* filter, the probability density of $\overline{\Phi_{xx}}(\omega)/\Phi_{\overline{xx}}(\omega)$ should be known. (For reasons of brevity the argument ω will frequently be dropped from now on). Strackee and Cerri (1977) have shown, that when the additive noise is normally distributed and white, the joint probability density of

$$\nu = \frac{N\overline{\Phi_{xx}}}{\Gamma_{nn}}$$

and (3.1)

$$\tau = \frac{N\Phi_{\overline{xx}}}{\Gamma_{nn}}$$

equals

$$f(\nu,\tau) = \frac{\{\nu-\tau\}^{N-2}\ e^{-N\Theta-\nu}\ I_0(2\sqrt{N\Theta\tau})}{(N-2)!} \qquad (0\leq\tau\leq\nu) \tag{3.2}$$

where Θ is the SNPDR as defined by (2.12) and I_0 represents the modified Bessel function of order zero.

The probability density f(H) of the *a posteriori* estimated filter is obtained in the following way. From (2.16) and (3.1) it follows

$$H = \frac{N}{N-1}\left\{1 - \frac{\nu}{N\tau}\right\} \quad \text{or} \quad \nu = \{N - (N-1)H\}\tau \tag{3.3}$$

Thus (Papoulis, 1965)

$$f(H) = (N-1)\int_0^\infty \tau\ .\ f(\{N - (N-1)H\}\tau,\tau)d\tau \tag{3.4}$$

Substitution of (3.2) into (3.4) and integration yields (Gradshteyn and Ryzhik, 1965; p. 720 and 1059):

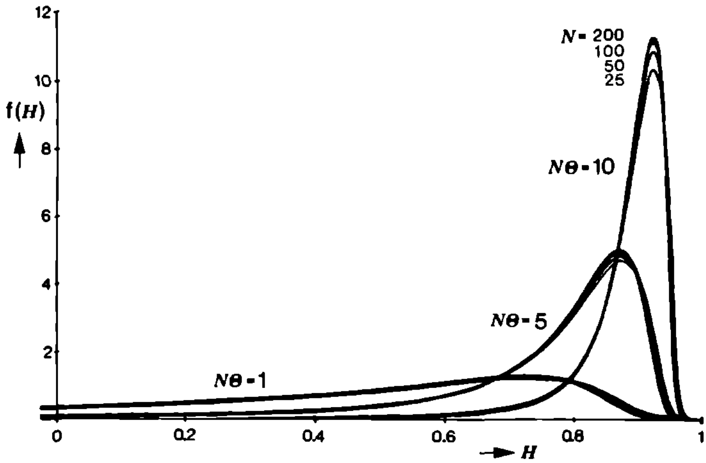

Figure 2.1

Probability density function of the values that are assumed by the estimated a posteriori *"Wiener" filter, as a function of the SNPDR after averaging NΘ (see text) and the number of stimulus repetitions N. Only the range that is physically meaningful is shown ($0 \leqq H \leqq 1$). With decreasing NΘ, however, the probability density flattens out over negative values*

$$f(H) = \left\{\frac{N-1}{N}\right\}^N e^{-N\Theta} \frac{\{1-H\}^{N-2}}{\left\{1-(\frac{N-1}{N})H\right\}^N} \cdot M\left(N, 1;\ \frac{\Theta}{1-(\frac{N-1}{N})H}\right) \qquad (H \leqq 1) \tag{3.5}$$

where the symbol M represents Kummer's function (see e.g. Abramowitz and Stegun, 1970; p. 504). In particular:

$$M(N,1;u) = 1 + Nu + \frac{N(N+1)u^2}{(2!)^2} + \frac{N(N+1)(N+2)u^3}{(3!)^2} + \ \ldots.. \tag{3.6}$$

(A similar expression as (3.5) has been found by Strackee and Cerri (1977) for the probability density of ν/τ).

The density f(H) has been plotted in Figure 2.1 for some typical values of NΘ (i.e. the signal-to-noise power density ratio after averaging) and N (the number of stimulus repetitions).

In the first place it is interesting to note that for given NΘ the probability density depends only very slightly on N. A dependence stems from the fact that the spectrum $\overline{\Phi_{xx}}$ is composed of an average of N spectra of individual ensemble elements and hence its variability reduces with increasing N. The variability of the spectrum $\Phi_{\overline{xx}}$, however, depends solely on the SNPDR after averaging ($N\Theta$). In the statistical behaviour of the filter this variability becomes more and more the dominating factor as $N\Theta$ decreases, and consequently the dependence on N reduces accordingly.

Secondly it is obvious from Figure 2.1 that the probability density becomes very unsymmetrical with decreasing $N\Theta$ and H may even become negative, which is physically not justifiable. Strackee and Cerri (1977) have calculated the probability for a negative filter value as a function of $N\Theta$. They found 63, 35, 17 and 1% for $N\Theta = 0,1,2$ and 5 respectively. This finding has brought them, following Walter (1969), to the conclusion that the averaging procedure which precedes the application of the filter, should be continued until no more negative values for H do occur. This is equivalent to stating that the signal-to-noise power density ratio $N\Theta$ should be large enough to reduce the probability for a negative value sufficiently. The higher the SNPDR, however, the less effective the filter becomes, as has been shown in the previous section. An alternative approach to this problem, which considerably reduces the occurrence of negative values, will be explained in the next section.

As a further analysis we consider the first two moments of the estimated *a posteriori* filter values, as a function of $N\Theta$ i.e. the SNPDR after averaging. Since it is impossible to obtain these two moments in a closed expression, they have been approximated by numerical integration, using (3.5). For improved accuracy, Kummer's function $M(N,1;u)$ is evaluated by a series expansion in Bessel functions (Abramowitz and Stegun, 1970; p. 506), and double precision techniques are used. The integration is terminated at H_{min}, such that

$$\int_{H_{min}}^{1} f(H)dH > 0.9999 \qquad (H_{min} \leq 1) \tag{3.7}$$

Results of this numerical procedure are shown in Figure 2.2 (N fixed to 100). For $N\Theta$ lower than about 6 the variance increases sharply (and tends to infinity as can be shown theoretically). Obviously there is also a (negative) bias which becomes progressively larger with decreasing $N\Theta$. The mean value of the estimated filter even becomes negative for $N\Theta$ lower than 2.5, and tends to minus infinity for $N\Theta$ approaching zero. Thus it is concluded, that by replacing the theoretical spectra Γ_{xx} and $\Gamma_{\overline{xx}}$ in (2.8) by their estimates $\overline{\Phi}_{xx}$ and $\Phi_{\overline{xx}}$ one obtains an estimated filter transfer function that has a progressively larger bias and variance as the signal-to-noise power density ratio decreases.

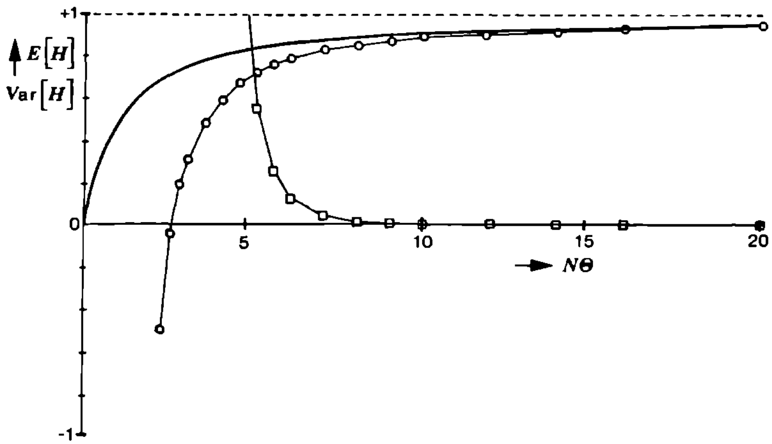

Figure 2.2

Approximate mean value (circles) and variance (squares) of the estimated a posteriori *"Wiener" filter vs. the SNPDR after averaging NΘ (N=100). For comparison, the value of the theoretical* a posteriori *Wiener filter is also shown (solid curve)*

A similar problem occurs in the estimation of frequency transfer functions using cross spectral analysis. In the system shown in Figure 2.3, the transfer function $H(\omega)$ may be estimated by

$$\hat{H}(\omega) = \frac{\Phi_{xy}(\omega)}{\Phi_{xx}(\omega)} \tag{3.8}$$

where $\Phi_{xy}(\omega)$ is the cross-spectrum between input and output. In general, such an estimator is inconsistent, unless some form of smoothing is applied to both the cross- and the auto-spectrum. It is precisely this smoothing procedure, which should also be applied in estimating the *a posteriori* filter transfer function in order to reduce both bias and variance to an acceptable level, which is shown in the next section.

Finally, it appears from the literature, that with the exception of Strackee and Cerri (1977), most authors have ignored the problem of occurrence of negative values in the estimated filter by defining a "clipped" a.p.w.f. according to

$$\begin{aligned} \hat{H}^c(\omega) &= \hat{H}(\omega) \qquad && \hat{H}(\omega) \geq 0 \\ &= 0 && \hat{H}(\omega) < 0 \end{aligned} \tag{3.9}$$

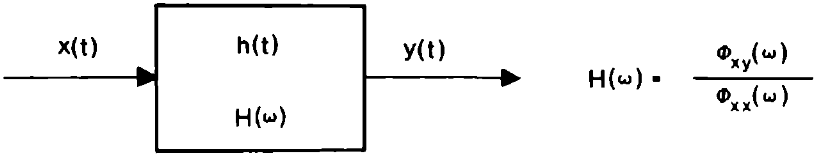

Figure 2.3
Estimation of a filter transfer function

In view of the foregoing, however, such filter behaviour is far from optimal in those frequency ranges where the SNPDR is low. In that case not only is the filtering too rigorous, due to the negative bias, but moreover the results are quite unpredictable due to the large variance which may cause large fluctuations in the estimated filter values. Therefore, the application of such a filter should be discouraged, in spite of the fact that it may result in a smaller mean-square error to that achieved by averaging alone (cf. Section 6).

4. Analysis of the Smoothed *a posteriori* Filter

Starting from (2.16) we now define the smoothed transfer function $\hat{H}^s(\omega)$ as

$$\hat{H}^s(\omega) = \frac{N}{N-1}\left\{1 - \frac{1}{N}\,\frac{\overline{\Phi^s_{xx}}(\omega)}{\Phi^s_{\overline{xx}}(\omega)}\right\} \tag{4.1}$$

where $\overline{\Phi^s_{xx}}(\omega)$ and $\Phi^s_{\overline{xx}}(\omega)$ represent smoothed spectral estimates:

$$\overline{\Phi^s_{xx}}(\omega) = \int_{-\infty}^{\infty} W_1(\xi)\cdot\overline{\Phi_{xx}}(\omega-\xi)d\xi \tag{4.2}$$

$$\Phi^s_{\overline{xx}}(\omega) = \int_{-\infty}^{\infty} W_2(\xi)\cdot\Phi_{\overline{xx}}(\omega-\xi)d\xi \tag{4.3}$$

and $W(\xi)$ is one of the common spectral windows (cf. Jenkins and Watts, 1968) which are often defined in the time domain. Rather arbitrary we choose the Tukey window:

$$\begin{aligned} w(u) &= \frac{1}{2}\left\{1 + \cos\left(\frac{\pi u}{M}\right)\right\} && |u| \leq M \\ &= 0 && |u| > M \end{aligned} \qquad M \leq T \tag{4.4}$$

Its spectral pendant follows from

$$W(\xi) = \int_{-\infty}^{\infty} w(u)e^{-j2\pi\xi u}du \tag{4.5}$$

Usually the choice of the truncation point M is a compromise between the bias and variance of the estimator. More precisely this compromise depends on the degree of smoothness of the theoretical spectra $\Gamma_{\overline{xx}}(\omega)$ and $\Gamma_{xx}(\omega)$. This choice also depends on the degree of smoothness which is actually required. In particular this applies to the spectrum $\overline{\Phi_{xx}}$, which in general is already smoothed due to the fact that it is an average of individual power density spectra. Therefore the windows appearing in (4.2) and (4.3) need not necessarily have the same truncation points. For convenience, however, in the following we will assume that $W_1(\xi) = W_2(\xi)$.

Using a Tukey window, the proportional reduction in variance of the spectra equals (Jenkins and Watts, 1968):

$$\frac{\mathrm{Var}\{\Phi^s_{xx}\}}{\Gamma^2_{xx}} = \frac{3M}{4T} \tag{4.6}$$

while the bandwidth of this window is given by

$$b = \frac{4}{3M} \ [\mathrm{Hz}] \tag{4.7}$$

As a rule of thumb the truncation point M may be chosen such that the bandwidth of the window is of the same order as the width of the narrowest significant peak in the spectrum.

For an analysis of the smoothed *a posteriori* filter we now proceed as follows. In view of the difficulties that arose in the previous section concerning the statistical analysis of the estimated filter, we restrict ourselves to an analysis of the first two moments, which will be estimated by a simulation. As a further simplification we will assume that the spectra of signal and noise, and thus the spectra Γ_{xx} and $\Gamma_{\overline{xx}}$ are flat. In that case it can be shown (Jenkins and Watts, 1968) that the smoothed spectral estimates $\overline{\Phi^s_{xx}}$ and $\Phi^s_{\overline{xx}}$ are unbiased.

Flat spectra are obtained by using a discrete "delta function" in the time domain, perturbed with additive "white" noise. After signal and noise have been averaged 100 times (N=100) a smoothed transfer function is calculated. This procedure is repeated to make up an ensemble of transfer functions, each function based on entirely different noise records. From this ensemble the mean and variance of the filter values are computed. It should be noted that mean and variance are independent of frequency, due to the fact that the theoretical spectra are flat.

Figure 2.4. shows the estimated mean value of the smoothed filters vs NΘ

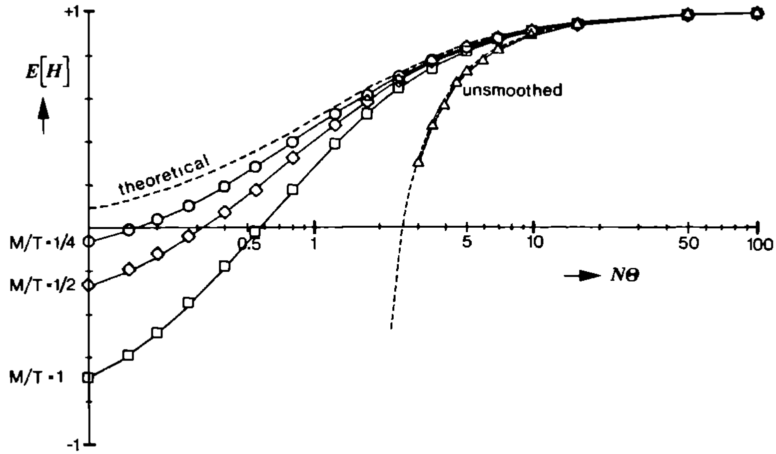

Figure 2.4
Estimated mean values of smoothed and unsmoothed a posteriori *"Wiener" filters as obtained by simulation vs. the SNPDR after averaging NΘ (N=100, solid lines). Smoothing is performed using a Tukey window with truncation point M. For comparison, the values of Figure 2.2 (for the theoretical and unsmoothed filter resp.) are also shown (broken lines)*

(on a log coordinate) for various truncation points M of the Tukey window, as compared with the value of the theoretical filter and the estimated mean value of the unsmoothed filter, based on the same simulation. For the latter filter, the results of Figure 2.2 (based on numerical integration of the theoretical density function) are also plotted. Obviously these results agree very well with the simulated values.

In Figure 2.5 the estimated variance of the smoothed filters is plotted, again for various truncation points M, as well as the variance of the unsmoothed filter, based on the same simulation. For comparison, the variance of Figure 2.2, which is slightly larger than the values obtained by simulation, is also shown. In general it is concluded from these figures, that a drastic improvement in the filter characteristics is obtained. This is true even with the smallest possible amount of smoothing (M/T=1), corresponding with the well known "Hanning" window i.e. a convolution of the spectra with a three point window having weights of $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{4}$. Under the condition that the underlying spectra are flat, the bias and variance of the smoothed filter become smaller with decreasing truncation point M. Ultimately, the influence of the truncation

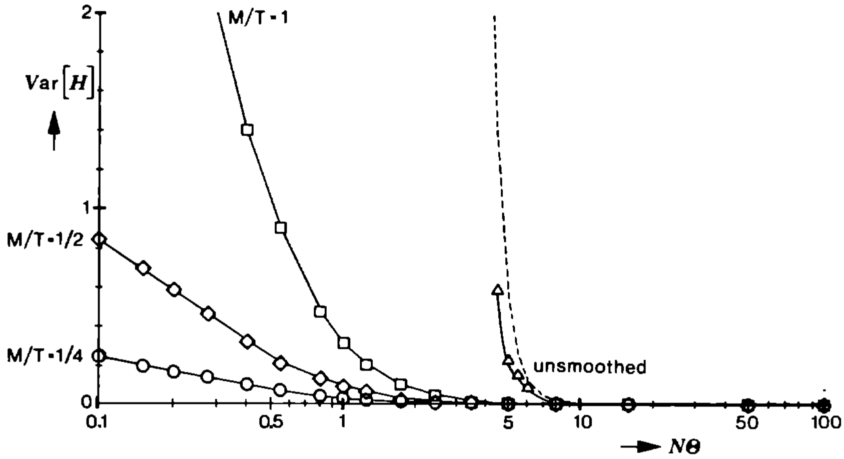

Figure 2.5

Estimated variance of smoothed and unsmoothed a posteriori *"Wiener" filters, as obtained by simulation vs. the SNPDR after averaging NΘ (N=100, solid lines). Smoothing is performed using a Tukey window with truncation point M. For comparison, the variance of Figure 2.2 (unsmoothed filter) is also plotted (broken line)*

point on bias and variance becomes linear, similar to (4.6). More precisely, it can be shown (de Weerd et al, 1979) that if the joint probability density function of $\Phi^s_{\overline{xx}}$ and $\overline{\Phi^s_{xx}}$ is concentrated near its center of gravity (which is the case after sufficient smoothing and not too low SNPDR, e.g. M/T=$\frac{1}{4}$ and NΘ >$\frac{1}{2}$) the bias and variance can be approximated by

$$\mathrm{Bias}\{\hat{H}^s(\omega)\} \cong -\frac{3M}{4T}\frac{2N\Theta+1}{\{N\Theta+1\}^3} \tag{4.8}$$

and

$$\mathrm{Var}\{\hat{H}^s(\omega)\} \cong \frac{3M}{4T}\frac{2N\Theta+1}{\{N\Theta+1\}^4} \tag{4.9}$$

In practice, the underlying spectra $\Gamma_{\overline{xx}}$ and Γ_{xx} are usually not flat, and a small value of M may cause considerable bias in the estimated spectra $\Phi^s_{\overline{xx}}$ and $\overline{\Phi^s_{xx}}$, which in turn will produce an additional bias in the filter $\hat{H}^s$.

In evoked potential analysis it is very difficult, as a matter of fact, to decide upon a value for M which is an acceptable compromise in all applications. Empirically a choice M/T=$\frac{1}{4}$ seems realistic and will be used in the remainder of this article.

Even though the smoothed a.p.w.f. closely approximates the theoretical filter, it still may happen that negative values of the transfer function occur, both due to the negative bias and as a consequence of the remaining variability. However, due to the large reduction in bias and variance these negative values do occur at more than 10 times lower SNPDR values. In this region the theoretical transfer function is relatively small already and therefore a clipping procedure as discussed in the previous section, which is still a necessary intervention, will introduce considerably smaller errors.

A final remark concerns a possible refinement of the method, as outlined in this section. This refinement applies in particular to the field of evoked potentials and is based on the observation that with increasing frequency, the SNPDR becomes lower and the spectral peaks wider. Therefore it may be argued that the amount of spectral smoothing should enlarge with frequency. This may be realized by applying a smoothing window of *proportional*, rather than *constant*, bandwidth.

5. Practical Aspects in the Application of Digital "Wiener" Filtering

In this section we pay attention to some practical problems that arise when *a posteriori* "Wiener" Filtering is applied in discrete time. It should be emphasized, that unless special precautions are taken, these problems may introduce considerable errors into the filtering procedure.

5.1 Aliasing

Aliasing is a notorious problem in digital signal processing, usually caused by undersampling of the original analog waveform. It results in spectral distortion, which in the application of a.p.w.f. produces an incorrect filter transfer function as well as improper filtering. In the analysis of evoked potentials, aliasing may be avoided effectively by means of analog filtering prior to AD Conversion and by choosing the sampling rate sufficiently high.

5.2 Leakage

Leakage is a fundamental problem in the Fourier Analysis of signals of a finite duration. The finite observation time may be thought of as multiplying the sampled signal with a rectangular time window, which in turn is equivalent to convolving the spectrum of the signal with a sinc function in the frequency domain. The large side lobes of this function cause spectral values at frequencies distant from f to contribute significantly to the value at frequency

f. This phenomenon is called (spectral) leakage. The amount of leakage can be reduced by applying a time window to the original signal, which has smaller side-lobes in the frequency domain. This procedure is usually referred to as "tapering". A commonly used technique is to apply a Tukey window (expression (4.4)) to approximately the first and last 10% of the signal, with a weight of unity in between.

5.3 Periodic Convolution

The occurrence of periodic or circular convolution in the time domain when multiplying two discrete Fourier transforms (DFT's) in the frequency domain is a problem which, in our experience, is very often overlooked. The nature of this problem lies in the fact that a time series of N samples which is Fourier transformed, results in a periodically extended spectrum with period N. Exactly the same reasoning holds true for the reverse, i.e. when a spectrum is Fourier transformed into a time series. The multiplication of two such spectra corresponds with a circular convolution, which can be regarded as a convolution, where samples from a sequence that are shifted out on one end of the range, are shifted in again into the other end (Gold and Rader, 1969). This difficulty can be overcome by augmenting each time series with sufficient zero values, such that there are no more interfering intervals.

To allow the use of the Fast Fourier Transform, the modified sequence length should again be adjusted to a power of 2. More precisely, let K_1 and K_2 be the sample size of both sequences respectively, with K_1 and K_2 powers of 2. Let the augmented series have a length K_3, then

$$\begin{aligned} K_3 &= 2K_1 \quad \text{if} \quad K_1 \geq K_2 \\ &= 2K_2 \qquad\quad K_1 < K_2 \end{aligned} \tag{5.1}$$

The foregoing problem arises when multiplying the transfer function of the a.p.w.f. with the DFT of the averaged signal, according to (2.17). Therefore, it is necessary to augment the averaged time signal with a number of zero values that corresponds with the expected length of the impulse response of the *a posteriori* Wiener filter. The problems associated with periodic convolution emerge also in smoothing the power spectra $\Phi_{\overline{xx}}$ and $\overline{\Phi_{xx}}$. The convolutions that appear in (4.2) and (4.3) are, in practice, evaluated by a multiplication in the time domain, followed by a FFT into the frequency domain again. In this case it is advantageous that the length of the Tukey window in the frequency domain, which can be determined from

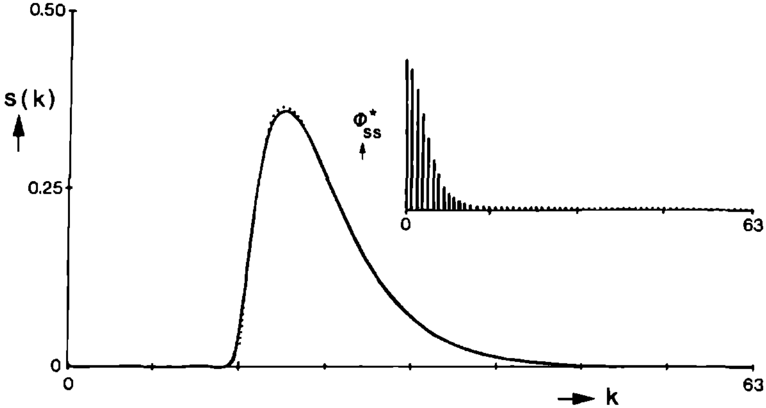

Figure 2.6
Original (dotted line) and modified waveform (solid line) of the signal used in the simulation. The modified waveform is the result of multiplying the original interpolated spectrum (Φ^*_{ss}, inset) with a cosine window in order to obtain a somewhat "smoother" leading edge

$$W(\omega) = M \frac{\sin(\omega M)}{\omega M} \left\{ \frac{1}{1 - \left(\frac{\omega M}{\pi}\right)^2} \right\} \tag{5.2}$$

is known exactly as a function of the truncation point M. This implies that the minimum number of zero values that should be added in the frequency domain is known beforehand.

6. A Case Study

In this section we analyse the performance of the a.p.w.f. in more detail, using an artificial signal and additive white noise. In particular, the results of ensemble averaging, *a posteriori* clipped-, smoothed-, and theoretical "Wiener" Filtering will be compared as a function of the signal-to-noise ratio. This case study is performed, since most of the expressions as derived in the previous sections can only be evaluated on the basis of a specific example. This also implies, however, that the results obtained are of limited value and cannot simply be generalized to arbitrary cases.

6.1 Simulated Signal and Noise

As a starting point we take the same signal as used by Strackee and Cerri (1977) i.e.

$$s(k) = k\alpha e^{-k\alpha} \qquad \begin{array}{l} \alpha = 0.25 \\ k = 0,1.....63 \end{array} \tag{6.1}$$

In order to obtain a somewhat "smoother" leading edge of this signal, in the simulations a low-pass filtered version of (6.1) is used. The filtering is performed by multiplying the DFT of (6.1) by a cosine window. Figure 2.6 shows the result of this procedure. (In this and the remaining figures, the signal has been shifted on the k-axis for reasons of clearness).

The (white) noise is software generated by using realisations of a random variable with gaussian distribution and zero expectation.

6.2 Computational Procedures

At various signal-to-noise ratios the signal and noise have been averaged up to 1000 times. After every tenth (in the range N=1-100) or every hundredth (in the range N=100-1000) record three *a posteriori* filters were calculated:

- the clipped filter $\hat{H}^c(\omega)$ (cf. 3.9)
- the smoothed filter $\hat{H}^s(\omega)$ (cf. 4.1)
- the theoretical filter $H(\omega)$ (cf. 2.7)

The latter filter is computed from the known power density spectrum of the signal s(k) (cf. 6.1) and an estimate of the theoretical noise spectrum, based on an average of 10^4 individual noise power density spectra. The three filters are applied to the averaged signal, whereafter the respective mean-square errors are calculated, according to (2.9). Due to the statistical nature of the noise, these mean-square errors may vary considerably for different noise ensembles. To bring out the underlying characteristics of the methods the entire averaging and computational procedure is repeated 100 times for different noise realizations, whereafter the respective mean-square errors (MSE) are averaged. Finally, to enable an easy comparison of the average estimation errors at different signal-to-noise ratios (SNR), these errors are normalized with respect to the SNR before averaging, i.e.:

$$e(N) = \rho\bar{\varepsilon}(N) \tag{6.2}$$

where $\bar{\varepsilon}(N)$ denotes the averaged MSE and the signal-to-noise ratio ρ is defined as

$$\rho = \frac{\int \Gamma_{ss}(\omega)d\omega}{\int \Gamma_{nn}(\omega)d\omega} \tag{6.3}$$

e(N) will be referred to as the normalized AMSE.

6.3 Normalized Average Mean-Square Errors

Figure 2.7 shows, on log-log coordinates, normalized AMSE curves vs. the number of stimulus repetitions N at three typical initial SNRs, i.e. ρ = -25, -15, and -5dB respectively. The range of signal-to-noise ratios has been chosen such that the three figures are partially overlapping as far as the SNR after averaging is concerned. More precisely, the SNR after averaging and consequently the AMSE curve e_A in the range N=100-1000 corresponds with that in the range N=10-100 of the next lower figure. From Figure 2.7 it becomes obvious that this correspondence applies equally well to the curves following *a posteriori* filtering, provided that N is not too small (N>20). This implies that the performance of the filter depends mainly on Nρ i.e. the signal-to-noise ratio after averaging (or before filtering), rather than on the number of stimulus repetitions. The restriction to large N stems from the fact that for small N the variance of the averaged power spectrum $\overline{\Phi^s_{xx}}$ cannot be neglected as compared to the variance of the power spectrum of the average $\Phi^s_{\overline{xx}}$ and contributes significantly to the variability of the estimated filter. These findings are in agreement with the theoretical arguments that have been discussed in Section 3.

A second result that follows from Figure 2.7 is that all *a posteriori* filters, even the somewhat unrealistic clipped filter, produce in expectation a reduction of the mean-square error. This remains true, not only in expectation, but also when applying a.p.w.f. to an individual ensemble average and regardless of the smoothing technique that has been used. Simulations with different signals, including signals resembling evoked potentials, and noise have shown similar results. Therefore it may be concluded that under general conditions application of an *a posteriori* "Wiener" filter cannot lead to a larger (mean-square) estimation error.

The relative performance of the different methods is also apparent from Figure 2.7. With increasing signal-to-noise ratio the AMSE of the clipped filter approaches that of the smoothed filter, which in turn approximates the theoretical minimum value, indicating that the transfer function of both filters becomes closer to the theoretical optimal one. At the same time, however, these AMSE curves show a decreasing distance to the 1/N line, which represents the AMSE after ensemble averaging. This finding illustrates that although the transfer function is better estimated with increasing SNR, the filter becomes less effective at the same time. This behaviour is more explicitly demonstrated in Figure 2.8, which shows the ratios $e_A(N)/e_S(N)$ and $e_A(N)/e_T(N)$ vs. the signal-to-noise ratio after averaging (or before filtering) for N relatively large (N>20). These ratios represent the decrease in estimation error due to *a pos-*

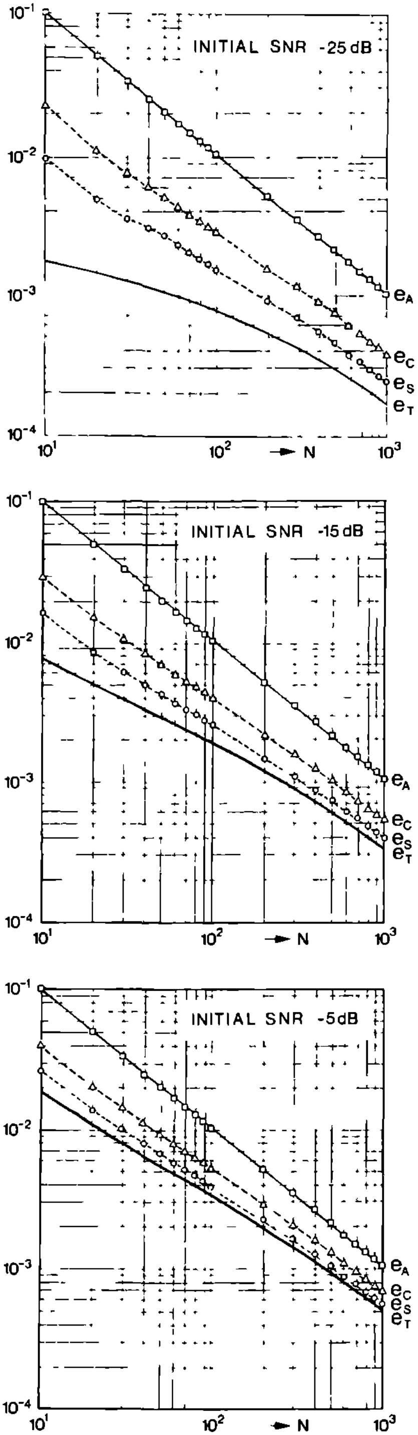

Figure 2.7
Normalized average mean-square errors (AMSE) vs. the number of stimulus repetitions N for the signal of Figure 2.6, perturbed with additive white noise at three different initial signal-to-noise ratios ρ. The error curves refer to e_A*: ensemble averaging;* e_C*: clipped a.p.w.f.;* e_S*: smoothed a.p.w.f.;* e_T*: theoretical a.p.w.f. It can be shown that the AMSE from an* a posteriori *"Wiener" filter (a.p.w.f.), which is linear and time-independent, lies in between the two solid lines* e_A *and* e_T*, regardless of the smoothing technique that has been used. Note that with increasing signal-to-noise ratio, the error curves converge to the 1/N line, indicating that* a posteriori *filtering becomes less effective*

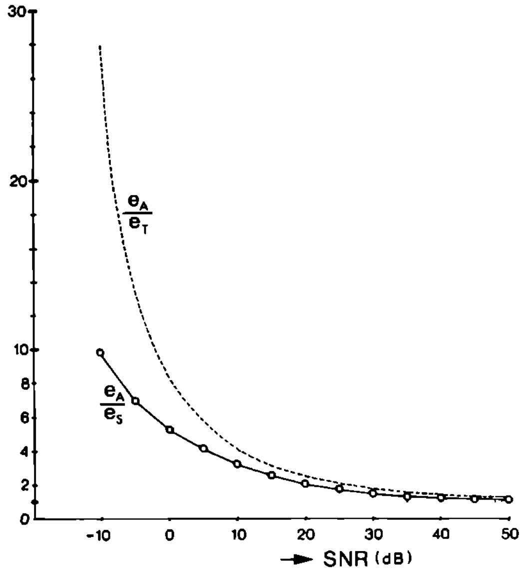

Figure 2.8
Relative decrease in estimation error, due to theoretical a.p.w.f. (broken line) and smoothed a.p.w.f. (circles) vs. $N\rho$, i.e. the signal-to-noise ratio after averaging (before filtering). Definitions of the errors e_A, e_T and e_S as in Figure 2.7. These curves apply exclusively to the specific example of the case study and cannot be generalized to arbitrary cases

teriori smoothed and theoretical filtering respectively. Clearly the reduction in the AMSE due to smoothed a.p.w.f. as compared to averaging alone, becomes progressively larger with decreasing SNR, but also progressively smaller than is expected theoretically. Since the AMSE after ensemble averaging $e_A(N)$ is proportional to the number of stimulus repetitions N, it follows that the ratios of Figure 2.8 may equally well be interpreted as the possible reduction in the number of stimuli vs. the SNR after averaging. This reduction is defined as the ratio of the number of stimuli using ensemble averaging and ensemble averaging followed by a.p.w.f. such that both estimates have equal AMSE. The usefulness of such an interpretation is discussed in more detail in the next section.

6.4 Filtered and Averaged Waveforms

Figure 2.9 shows typical waveforms that result from application of the different filters. SNR values are the same as in Figure 2.7. In the upper traces (a) the averages of 100 records are plotted, together with the original waveform (dotted line) for comparison. The lower traces (b-d) show the waveforms after application of clipped, smoothed and theoretical *a posteriori* filtering respectively. Clearly, the waveforms become smoother in this order. However, in comparing the waveforms at different SNR some essential differences appear. In I (Figure 2.9, left) the SNR *after* averaging equals approximately -5dB. At such a low SNR even in the output of the theoretical filter (trace Id) the

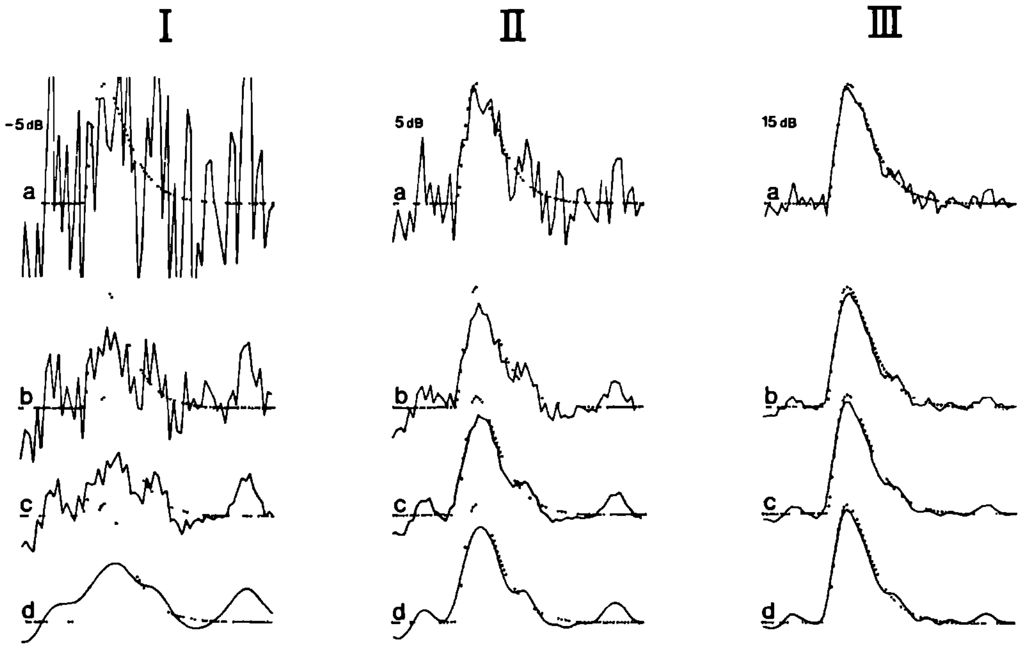

Figure 2.9I-III
Typical examples of averaged and a posteriori *filtered waveforms, starting from the signal of Figure ?.6 (dotted lines), perturbed with additive white noise, at same initial signal-to-noise ratios as Figure 2.7* (I: -25dB; II: -15dB; III: -5dB). a: *average of 100 records, SNR is improved with approximately 20 dB;* b: *waveform after application of clipped a.p.w.f. to the average of* a; c: *same for smoothed a.p.w.f.,* d: *same for theoretical a.p.w.f. Note that in spite of a considerable reduction in estimation error, at low SNR a.p.w.f. becomes ineffective due to the large distortion that emerges* (I). *At higher SNR, a.p.w.f. effectively leads to an improved estimate (*II *and* III*)*

original waveform cannot be recognized. This example illustrates that in spite of a reduction in the estimation error of 14 (cf. Figure 2.8), even theoretical filtering fails due to the large distortion that emerges. Indeed, application of the filter at this low SNR could lead to serious misinterpretations dictated by the suggestively smooth waveform. In II (Figure 2.9,-middle) the situation is somewhat different. The broad waveform of the signal is already

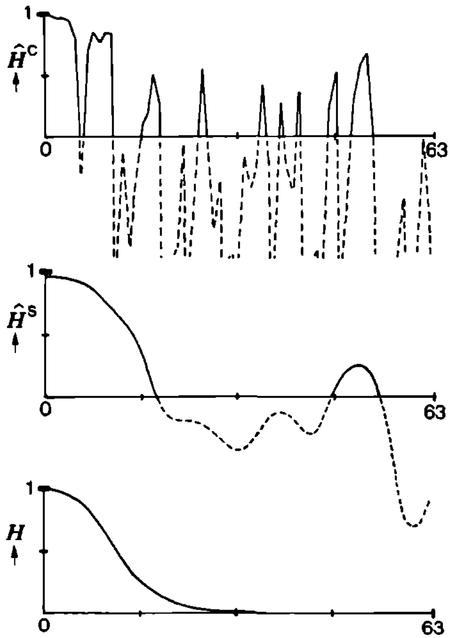

Figure 2.10
Typical example of a posteriori *filter transfer functions, corresponding with the waveforms* IIb,c,d *of Figure 2.9. The filters are based on an ensemble of 100 records, SNR before filtering +5dB.* $\hat{H}^c$*: transfer function of the* clipped *(unsmoothed) filter.* $\hat{H}^s$*: same for* smoothed *a.p.w.f. H:* theoretical *transfer function based on* a priori *knowledge of signal and noise spectra. Clipped filter values are represented by the broken lines*

recognized in the ensemble average, although large noise components remain present. According to Figure 2.8 a reduction of 4 (smoothed a.p.w.f.) or 6 (theoretical a.p.w.f.) can be achieved at this SNR. From the filtered waveforms it may be concluded that, in spite of a significant distortion, the recognizability of the original signal is greatly enhanced. Therefore, at these SNR values the method can be applied successfully to improve the estimate. By inspection of the waveform of IIb, this seems to be true for the unsmoothed a.p.w.f. as well. It should be recalled, however, that this filter may produce inconsistent results, which is not obvious from a single observation. Yet this can be made plausible by inspection of Figure 2.10, where the transfer functions corresponding with the waveforms of Figure 2.9II(b-d) are plotted. The transfer function of the clipped filter shows large random fluctuations, not only in the stopband but also in the passband, where the SNPDR is larger than approximately 2.5. Likewise, the filtered signal may vary accordingly.

Returning to Figure 2.9, the original signal is easily recognized in the averaged waveform in III (right), which corresponds to a SNR of 15dB. *A posteriori* filtering still reduces the estimation error (with approximately 2-3), although it can be argued that such a procedure is redundant since the error after ensemble averaging is sufficiently small already. In that case, however, *a posteriori* filtering may effectively be used to reduce the number of stimulus

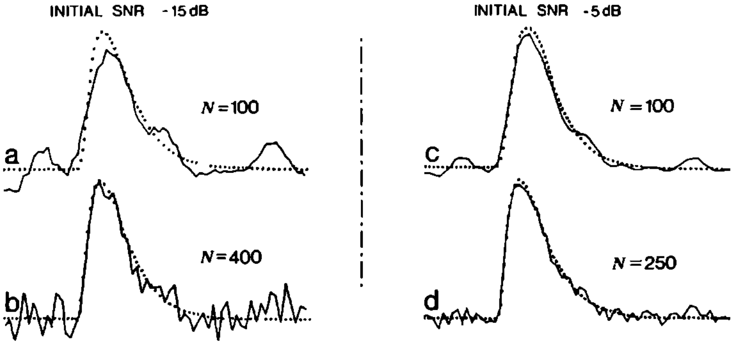

Figure 2.11a-d
Illustrative example to show the difference between averaging followed by smoothed a.p.w.f. and prolonged averaging at two signal-to-noise ratios. a: *waveform after averaging 100 records, followed by smoothed a.p.w.f.;* b: *waveform after averaging 400 records. Mean-square errors (MSE) of* a *and* b *are equal within 5%. Though waveform* a *is less noisy, this is at the price of systematic errors, i.e. a smaller peak amplitude and decreased leading signal slope.* c *and* d: *same as* a *and* b *at 10 times larger SNR. Systematic errors in* c *are almost absent. Note also that in this case the reduction in number of stimuli to obtain equal MSE is smaller*

repetitions.

The foregoing discussion concerning the use of a.p.w.f. in order to reduce the estimation error or alternatively reduce the number of stimuli is further illustrated by Figure 2.11, where in both upper traces the smoothed *a posteriori* filtered averages of Figure 2.9, IIc and IIIc have been redrawn (initial SNR −15 and −5dB respectively). The lower traces show waveforms having the same mean-square error (within 5%) as the corresponding upper waveforms, but obtained after prolonged averaging (N=400 and N=250 respectively, determined from Figure 2.7). Comparing the left two waveforms reveals that the filtered version shows systematic errors which are not apparent in the average with N=400. The distortion appears in a somewhat smaller amplitude and a decrease in signal slope. Though a reduction in the number of stimuli (from 400 to 100) is within the range of possibility, this is at the price of a significant distortion. On the other hand such a distortion is almost absent in the filtered waveform of Figure 2.11 right. Hence in this region of SNR values the applica-

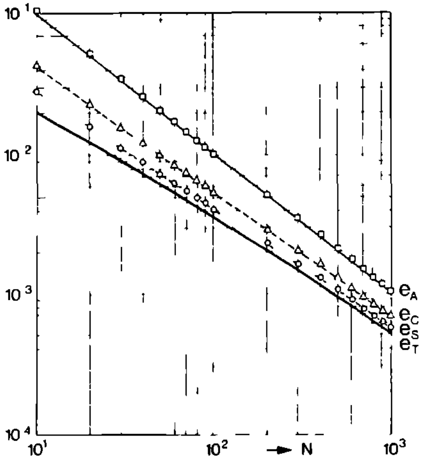

Figure 2.12
Normalized average mean-square errors (AMSE) vs. the number of stimulus repetitions N for the signal of Figure 2.6, slowly modulated in amplitude and perturbed with additive white noise (initial SNR -5dB). Signal amplitude was varied from record to record by multiplying with a random variable having a Gaussian distribution with mean 1 and standard deviation 0.4. Error curves are denoted as in Figure 2.7

tion of a.p.w.f. may indeed effectively reduce the number of stimuli.

6.5 Effects of Slow Amplitude Modulation

The question may arise as to whether the method of a.p.w.f. still remains applicable when the condition that the signal be invariant and the noise strictly additive is not met. This situation frequently arises in the field of evoked potentials where the amplitude of the signal may vary considerably; an effect which can be described as multiplicative noise of low frequency. In order to investigate this problem we define an ensemble of input signals $z_i(t)$:

$$z_i(t) = a_i s(t) + n_i(t) \qquad i = 1,2, \ldots\ldots N \tag{6.4}$$

where s(t) and n(t) are defined as in (2.1) and a_i represents a slow amplitude modulation:

$$\frac{1}{N}\sum_i a_i = 1$$

$$\text{and} \tag{6.5}$$

$$\frac{1}{N}\sum_i \{a_i - 1\}^2 = \sigma_a^2$$

The same computational procedures as described in Section 6.2 have been repeated for different SNR and modulation depth's σ_a.

Figure 2.12 shows a typical example of normalized AMSE curves that have been computed for an initial SNR = -5dB and σ_a = 0.4. Since the normalizing is performed with respect to the SNR before averaging, according to (6.2), Figure

2.12 is directly comparable with the lower Figure 2.7. Although it is difficult to discern any significant difference, a closer inspection reveals that the curves concerning the *a posteriori* filters are shifted slightly upwards. From this finding, which is consistent for various SNR and σ_a values, it is concluded that the relative performance of the filters is rather insensitive to slow amplitude variations in the signal.

7. Discussion

The technique of *a posteriori* "Wiener" filtering (a.p.w.f.) represents a linear, time-invariant filtering, which is therefore suited to improve the signal-to-noise ratio (SNR) of averaged evoked potentials that are of a nontransient character. In the following discussion we restrict ourselves for the time being to those cases for which this assumption is valid.

The improvement in the SNR by application of a.p.w.f. originates from a reduction in variability of the estimate, at the price of a higher distortion. In terms of estimation theory, this means that the variance in the estimate is reduced at the cost of an increased bias. Theoretically the compromise between these two is optimal for the Wiener filter in the sense that the mean-square error is minimized. In cases where the spectra of signal and noise do not overlap, the filtering is also perfect, i.e. the variance is suppressed entirely, while the estimate is unbiased. As far as evoked potentials are concerned, however, the spectra of signal and noise usually do show a large overlap. Moreover, in practice the theoretical least mean-square error is not reached, since the filter transfer function is itself an estimate.

Essentially, in ensemble averaging one does not have to compromise since, in general, averaging is an unbiased estimator (we will come back to this supposition later on). However, in practice there are many factors, such as slowly changing experimental conditions, habituation processes and arousal level that may degrade the averaging and set bounds to the duration of the experiment and thereby to the number of admissible stimulus repetitions. In such a situation one can be forced to choose between continued averaging or termination of the averaging and successive application of a.p.w.f. As a rule however, the technique of a.p.w.f. should not be considered primarily as a substitute for (continued) averaging, but merely as a method for improving the signal-to-noise ratio whenever further improvement by ensemble averaging is unlikely or impossible.

From the literature it may be concluded that the general praxis of a.p.w.f. is the application of an unsmoothed and clipped filter, as discussed in Section 3. It has been demonstrated that this indeed leads to a reduction in the estimation error. In spite of this lower error, however, such a filter is not consistent and suffers from a large negative bias and variance. As a consequence, application of this filter results in an unpredictable distortion especially in those frequency ranges where the signal-to-noise power density ratio (SNPDR) is relatively low. On the other hand, the estimation of the transfer function of the a.p.w.f. from spectra that have been adequately smoothed, reduces both the large variance and the negative bias, thereby extending the operating range of the filter significantly. This in turn results in a much better approximation to the theoretical filter transfer function which implies that the estimated waveform becomes more reliable and the estimation error accordingly lower.

The question whether or not a.p.w.f. is a useful technique in a particular application depends to a large extent on the signal-to-noise ratio (or more precisely: on the range of SNPDR values) of the averaged signal. From the theoretical point of view this is most easily demonstrated on the basis of a specific example. In the case study in Section 6, we may roughly differentiate between three regions of SNR values. In the first region (SNR < 0dB) the estimation error following a.p.w.f. becomes progressively smaller as compared with the error following ensemble averaging (cf. Figures 2.8 and 2.9). Yet the application of a.p.w.f. leads to an unacceptable distortion, although the resulting waveform may appear deceptively smooth and reliable. In the second region (0dB < SNR < 20dB), a.p.w.f. can be effectively applied in order to reduce the estimation error (with a factor 2-5), since the variability of the averaged waveform is reduced considerably whereas the distortion is still acceptable. In the third region (SNR > 20dB) the signal-to-noise ratio after averaging is sufficiently large already, and though application of a.p.w.f. still reduces the estimation error (with a factor 2 or less) the net effect is relatively small. Therefore, in this region, the method may well be used in order to reduce the number of stimulus repetitions.

It should be stressed that the actual SNR values which have been set forth in this example apply exclusively to the case study of the previous section and can in no way be generalized to arbitrary cases. Nevertheless a differentiation in three regions of application, though not as distinct as suggested above, can always be made.

In practice, the judgement upon the usefulness of a.p.w.f. should be approached more intuitively. Usually some knowledge regarding the approximate shape of the expected waveform is available. Then, as a rule of thumb, a.p.w.f.

is worth trying if the waveform can be reasonably recognized in the midst of the noise. If this is not the case, a.p.w.f. should not be applied, since misleading interpretation may be the result. When it is obvious, that the expected signal is much larger than the remaining noise, application of a.p.w.f. may eventually lead to a reduction in the number of stimulus repetitions.

Thusfar, we have assumed that ensemble averaging is an unbiased estimator, which increases the signal-to-noise power ratio proportional to the number of stimulus repetitions. This assumption is also essential in the theory of *a posteriori* filtering, as becomes obvious from (2.7). As long as the noise at different stimulus presentations is uncorrelated and the evoked potentials following subsequent stimuli do not overlap, this assumption does indeed remain valid. Ruchkin (1965) has shown what happens if these conditions are not met by describing the ensemble averaging as a linear filter procedure in the frequency domain. In his article it is concluded that when suitable *aperiodic* stimuli are used (e.g. having a uniform or exponential density), ensemble averaging does reduce the noise power with approximately 1/N in all but the very low frequency ranges. As an example, it follows that with an average interstimulus interval of 4s and a standard deviation of 0.6s (uniform density) or 2s (exponential density), the approximation to 1/N holds for frequencies not lower than ca. 1 Hz. This is about the lowest frequency that is still of interest in an evoked potential with a duration of one second or less.

Albrecht and Radil-Weis (1976) have argued that the factor 1/N, appearing in the *a posteriori* filter transfer function (cf. 2.7) should be replaced by an appropriate frequency weighing function, which depends on the probability density of the interstimulus intervals. In view of the foregoing example, however, it is concluded that such a modification will not contribute significantly to a better estimation procedure, provided that no periodic stimulation is used, which in our opinion should be a golden rule in the analysis of evoked potentials.

As stated before, the applicability of a.p.w.f. rests on the assumption that the signal is of an (almost) periodic, and the noise of a random stationary character. For most *transient* evoked potentials, the somatosensory evoked potential in particular, such an assumption is untenable. However, the technique of a.p.w.f. can be generalized to the technique of time-varying filtering, which solves the problems associated with transient signals to a large extent. The present study is also intended to form the basis for that method, which will be dealt with in future publications (de Weerd, 1981; de Weerd and Kap, 1981).

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Chapter 3

Estimation of Signal and Noise Spectra by Special Averaging Techniques with Application to *a posteriori* "Wiener" Filtering

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Biol. Cybernetics 32, 153-164 (1979)

Abstract

This paper deals with the problem of separating the spectra of signal and noise in ensembles where the signal can be considered as an invariant component and the noise as a stationary additive background. Several methods are discussed and compared on the basis of a statistical analysis of the first two moments of the estimators for signal and noise spectra. As a consequence a procedure is proposed which provides a flexible compromise between estimation accuracy and computational effort. The application of this procedure to *a posteriori* "Wiener" filtering is compared with a more common, but time consuming, technique.

1. Introduction

Ensemble averaging certainly represents one of the most common techniques in the analysis of electrophysiological data, not only in stimulus-response situations (evoked potentials), but also in cases where no stimulus or *a priori* trigger is available. In the latter case one should have the ability to derive a point of reference from the response itself.

As is well known, the technique of ensemble averaging is usually based on the assumption that each response or ensemble element may be thought of as a time-locked invariant signal and an additive stationary noise component which is not time-locked to the stimulus:

$$x_i(t) = s(t) + n_i(t) \qquad i = 1,2,\ldots\ldots,N \tag{1.1}$$

where s(t) represents the "real" signal, $n_i(t)$ a stationary noise process with zero mean and N the number of ensemble elements.

If s(t) and $n_i(t)$ are uncorrelated, ensemble averaging of $x_i(t)$ leads to:

$$\bar{x}_N(t) = s(t) + \frac{1}{N}\sum_{i=1}^{N} n_i(t) \qquad 0 \leq t \leq T \tag{1.2}$$

where T is the duration of the signal and the dash above the x represents ensemble averaging over the number of elements indicated by the suffix. Clearly, by ensemble averaging over a *finite* number of elements, it is not possible to *separate* the signal and the noise component completely, a problem that has occupied many researchers in the field of evoked potentials. This also implies that the use of (1.2) as such for the estimation of the signal power density spectrum would lead to a *biased* estimate, due to the remaining noise component.

In this paper several methods for obtaining *unbiased* estimates for both signal and noise spectra are discussed along with the advantages and disadvantages of each method in terms of estimation accuracy and computational cost.

The estimation of the spectra of signal and noise may be a goal in itself, but the spectral estimates can also be used to compute a "Wiener" filter. When applied to the ensemble average $\bar{x}_N(t)$ this should lead to a better estimate (in the mean-square error sense) of the real signal s(t). This idea and a computational procedure for the separation of signal and noise spectra was to our knowledge first proposed by Walter (1969) and has been discussed extensively in the recent literature (e.g. Doyle, 1975; Ungan and Basar, 1976; Albrecht et al., 1977; Strackee and Cerri, 1977; de Weerd and Martens, 1978). A statistical analysis of the spectral separation method, based on the first two moments, is presented in Sect. 2.

A disadvantage of this procedure, however, is the fact that it is relatively (computer-)time consuming and therefore not very well suited for multichannel applications in real time, or for extension to the estimation of time-varying spectra (de Weerd et al., 1977; de Weerd and Kap, 1981). From the computational point of view a better method is to estimate the signal and noise spectra by using an idea suggested by Schimmel (1967), i.e. the composition of an *alternate* average, in addition to the normal ensemble average. But the large savings in computation time are at the cost of an increased variance (Sect. 3).

In Sect. 4 it is shown that by introducing the concept of *subensemble averaging* both previous estimation methods can be generalized, whereby they become very similar. This generalization leads to effective estimation procedures, providing a flexible compromise between computation time and estimation accuracy. In the same section the choice of one particular method, i.e. estimation by ensemble and alternate subensemble averaging, is suggested, based on the notion that in practice the ensemble is rarely homogeneous. Finally, in

Sect. 5 the application of the newly proposed method to *a posteriori* "Wiener" filtering is compared with a technique that has been extensively discussed in the recent literature. This comparison is made by analysis of the first two moments of the estimated filter transfer function, based on an approximate statistical evaluation and on simulation studies.

2. Ensemble and Ensemble Spectra Averaging

The spectral estimation procedure discussed in this section was introduced by Walter in order to compute the transfer function of an *a posteriori* "Wiener" filter (Walter, 1969).
Essentially the technique is based on the composition of two power density spectra:
- a spectrum of the ensemble average according to (1.2), in which the noise power is reduced with a factor proportional to the number of ensemble elements
- an average of the spectra of individual ensemble elements, in which the noise power is not reduced.

More precisely, let the power density spectrum of $\bar{x}_N(t)$ be denoted by $\Phi_{\overline{xx}_N}(\omega)$. Then from (1.2) and assuming noise at different stimulus presentations to be uncorrelated (cf. de Weerd and Martens, 1978):

$$E\{\Phi_{\overline{xx}_N}(\omega)\} \triangleq \Gamma_{\overline{xx}_N}(\omega) = \Gamma_{ss}(\omega) + \frac{1}{N}\Gamma_{nn}(\omega) \tag{2.1}$$

Throughout this paper the symbol Γ will be used as the expectation of the power density spectrum Φ.
Thus $\Gamma_{\overline{xx}_N}$, Γ_{ss} and Γ_{nn} represent the theoretical power density spectra of the ensemble average, the signal and the noise respectively. Averaging the power density spectra of the individual ensemble elements leads to

$$E\{\overline{\Phi_{xx}}_N(\omega)\} \triangleq \Gamma_{xx}(\omega) = \Gamma_{ss}(\omega) + \Gamma_{nn}(\omega) \tag{2.2}$$

where Γ_{xx} represents the theoretical power density spectrum of x(t).

When the power density spectra of signal and noise are unknown, they can be estimated using (2.1) and (2.2):

$$E\{\hat{\Phi}_{ss}(\omega)\} \triangleq \Gamma_{ss}(\omega) = \frac{N}{N-1}\{\Gamma_{\overline{xx}_N}(\omega) - \frac{1}{N}\Gamma_{xx}(\omega)\} \tag{2.3}$$

and

$$E\{\hat{\Phi}_{nn}(\omega)\} \triangleq \Gamma_{nn}(\omega) = \frac{N}{N-1}\{\Gamma_{xx}(\omega) - \Gamma_{\overline{xx_N}}(\omega)\} \tag{2.4}$$

where $\hat{\Phi}_{ss}(\omega)$ and $\hat{\Phi}_{nn}(\omega)$ represent the estimated power density spectra of the signal and the noise respectively.

The variance of the estimated signal spectrum follows from

$$\begin{aligned} \mathrm{Var}\{\hat{\Phi}_{ss}(\omega)\} = (\tfrac{N}{N-1})^2 \Big(& \mathrm{Var}\{\Phi_{\overline{xx_N}}(\omega)\} + \frac{1}{N^2}\mathrm{Var}\{\overline{\Phi_{xx_N}}(\omega)\} + \\ & - \frac{2}{N}\mathrm{Cov}\{\Phi_{\overline{xx_N}}(\omega), \overline{\Phi_{xx_N}}(\omega)\}\Big) \end{aligned} \tag{2.5}$$

The variance and covariance terms that appear at the right hand side of this expression are evaluated in Appendix 1. Substitution of (A 10), (A 11), and (A 13) reduces (2.5) to

$$\mathrm{Var}\{\hat{\Phi}_{ss}(\omega)\} = \frac{\Gamma^2_{nn}(\omega)}{N(N-1)} + \frac{2}{N}\Gamma_{nn}(\omega)\Gamma_{ss}(\omega) \tag{2.6}$$

This result was derived earlier by Strackee and Cerri (1977) using the probability densities of the spectra $\Phi_{\overline{xx_N}}(\omega)$ and $\overline{\Phi_{xx_N}}(\omega)$.

From (2.6) it follows that for $N \gg 1$:

$$\mathrm{Var}\{\hat{\Phi}_{ss}(\omega)\} \cong \frac{\Gamma^2_{nn}(\omega)}{N^2} + \frac{2}{N}\Gamma_{nn}(\omega)\Gamma_{ss}(\omega) \tag{2.7}$$

The right hand term of this expression is identical to the expression for the variance of the spectrum $\Phi_{\overline{xx_N}}(\omega)$ [cf. (A 10)]. This implies that the variance of the signal spectrum according to (2.3) is *mainly* determined by the variance of the spectrum $\Phi_{\overline{xx_N}}$, while the variance of the spectrum $\overline{\Phi_{xx_N}}(\omega)$ is of virtually *no influence*.

The variance of the noise spectrum follows in a similar way as before:

$$\mathrm{Var}\{\hat{\Phi}_{nn}(\omega)\} = \frac{\Gamma^2_{nn}(\omega)}{N-1} \tag{2.8}$$

[This result is slightly different from that obtained by Strackee and Cerri (1977)].

The interpretation of (2.6) and (2.8) is simplified when we consider the relative variances:

$$\frac{\mathrm{Var}\{\hat{\Phi}_{nn}(\omega)\}}{\Gamma^2_{nn}(\omega)} = \frac{1}{N-1} \tag{2.9}$$

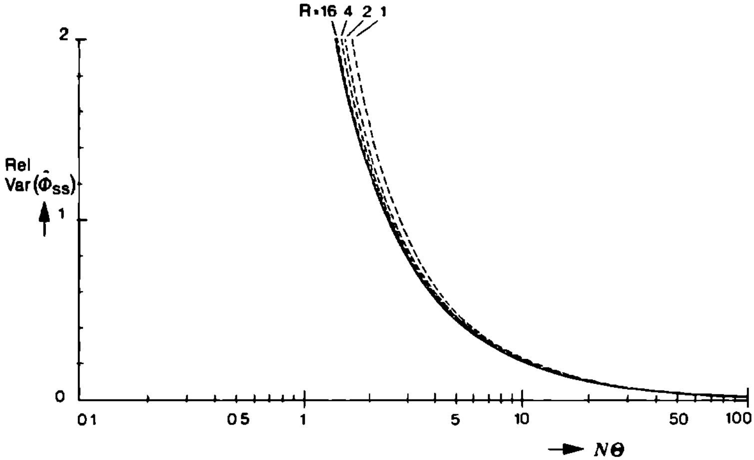

Figure 3.1
Relative variance of signal power density spectrum estimators vs. the SNPDR of the ensemble average NΘ (see text). Solid line: *estimation by ensemble and ensemble spectra averaging (Sect. 2).* Broken lines: *estimation by ensemble and* alternate *subensemble averaging (Sect. 4); the number of subensembles is given by R. The case R=1 corresponds to the estimation procedure as described in Sect. 3*

and

$$\frac{\mathrm{Var}\{\hat{\Phi}_{ss}(\omega)\}}{\Gamma^2_{ss}(\omega)} = \frac{1}{1-\frac{1}{N}} \cdot \frac{1}{(N\Theta(\omega))^2} + \frac{2}{N\Theta(\omega)} \qquad (2.10)$$

where

$$\Theta(\omega) = \frac{\Gamma_{ss}(\omega)}{\Gamma_{nn}(\omega)} \qquad (2.11)$$

In a previous article (de Weerd and Martens, 1978) $\Theta(\omega)$ has been defined as the *signal-to-noise power density ratio (SNPDR).*

Note that $\Theta(\omega)$ represents the SNPDR in a single ensemble element, while from (2.1) and (2.11) it follows that $N\Theta(\omega)$ represents the SNPDR in the ensemble average.

From (2.9) and (2.10) the following conclusions may be drawn:

- the relative variance of the estimated noise spectrum decreases almost proportionally with N; it is *independent* of the SNPDR

- the relative variance of the estimated signal spectrum depends almost solely on the SNPDR of the ensemble average NO (Figure 3.1, solid line). At low SNPDR values the variance increases sharply, as a consequence of the increasing variability of the spectrum $\Phi_{\overline{xx}_N}(\omega)$. The variance can be reduced either by averaging over a larger ensemble (increasing N, thus increasing NO), or by spectral smoothing techniques (cf. Jenkins and Watts, 1968; de Weerd and Martens, 1978).

3. Ensemble and Alternate Ensemble Averaging

As in the previous section we determine the spectrum of the ensemble average. But instead of computing the average of individual power density spectra we now compute in addition the spectrum of an *alternate* average.

The alternate average is given by

$$\tilde{x}_N(t) = \frac{1}{N}\sum_{i=1}^{N}(-1)^{i-1}(s(t) + n_i(t)) \qquad \text{(N even)} \tag{3.1}$$

i.e. the polarity of the signal $x_i(t)$ is changed with each new ensemble element. (The mark above the x will be used to indicate alternate ensemble averaging over the number of elements indicated by the suffix). The technique of alternate averaging was introduced earlier by Schimmel (1967), who called the result of this procedure the "($\pm$) Reference". As stated by Schimmel, the ($\pm$) Reference permits an estimation of the magnitude and statistical structure of the noise component that remains in the normal ensemble average [cf. (1.2)].

Indeed, by alternate addition and subtraction of ensemble elements, the signal s(t) cancels out, and the power density spectrum of the alternate average $\tilde{x}_N(t)$ is an estimate of the power density spectrum of the noise, reduced by a factor N. The proof is as follows.

Let the Fourier transform of $\tilde{x}_N(t)$ be denoted by $\tilde{X}_N(\omega)$ and its power density spectrum by $\Phi_{\tilde{x}\tilde{x}_N}(\omega)$. Then

$$\Phi_{\tilde{x}\tilde{x}_N}(\omega) = \tilde{X}_N(\omega)\tilde{X}_N^*(\omega) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}(-1)^{i+j-2}\cdot(S(\omega) + N_i(\omega))(S^*(\omega) + N_j^*(\omega)) \tag{3.2}$$

where $S(\omega)$ and $N_i(\omega)$ represent the Fourier transforms of the signal and the i^{th} noise ensemble element, and the starring denotes the complex conjugate.

Since N is even there appear as many positive as negative terms $S(\omega)S^*(\omega)$ in the summation. Taking expectations in (3.2) and assuming as before that signal and noise are uncorrelated it follows:

$$E\{\Phi_{\tilde{x}\tilde{x}_N}(\omega)\} \triangleq \Gamma_{\tilde{x}\tilde{x}_N}(\omega) = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N}(-1)^{i+j-2}\,E\{N_i(\omega)N_j^*(\omega)\} \tag{3.3}$$

If also the noise at different stimulus presentations is uncorrelated, the terms appearing in (3.3) are non zero for i=j only, in which case (3.3) reduces to

$$\Gamma_{\tilde{x}\tilde{x}_N}(\omega) = \frac{1}{N^2}\sum_{i=1}^{N}(-1)^{2i-2}\,E\{N_i(\omega)N_i^*(\omega)\} = \frac{1}{N}\,\Gamma_{nn}(\omega) \tag{3.4}$$

From (3.4) and (2.1) it follows for the estimators of the signal and noise spectra:

$$E\{\hat{\Phi}_{ss}(\omega)\} = \Gamma_{\overline{xx}_N}(\omega) - \Gamma_{\tilde{x}\tilde{x}_N}(\omega) \tag{3.5}$$

and

$$E\{\hat{\Phi}_{nn}(\omega)\} = N\,\Gamma_{\tilde{x}\tilde{x}_N}(\omega) \tag{3.6}$$

The computational advantages of this method are obvious. The relatively time consuming calculation of power density spectra is needed only twice [once for $\Phi_{\overline{xx}_N}(\omega)$ and once for $\Phi_{\tilde{x}\tilde{x}_N}(\omega)$] instead of N+1 times for the method described in the previous section [once for $\Phi_{\overline{xx}_N}(\omega)$, and N times for computation of the spectra of individual ensemble elements]. Moreover the change of input polarity, to make up the alternate average, is a simple operation which is easily realised.

There is, however, also a disadvantage of the method which appears from the calculation of the variances of the estimated spectra.

From (3.5) it follows:

$$\mathrm{Var}\{\hat{\Phi}_{ss}(\omega)\} = \mathrm{Var}\{\Phi_{\overline{xx}_N}(\omega)\} + \mathrm{Var}\{\Phi_{\tilde{x}\tilde{x}_N}(\omega)\} - 2\mathrm{Cov}\{\Phi_{\overline{xx}_N}(\omega),\ \Phi_{\tilde{x}\tilde{x}_N}(\omega)\} \tag{3.7}$$

In Appendix 1 it is proven that the last term equals zero. Substitution of (A 10) and (A 20) into (3.7) results in

$$\mathrm{Var}\{\hat{\Phi}_{ss}(\omega)\} = \frac{2\Gamma^2_{nn}(\omega)}{N^2} + \frac{2}{N}\,\Gamma_{nn}(\omega)\Gamma_{ss}(\omega) \tag{3.8}$$

while, for the variance of the noise spectrum it follows with (A 20):

$$\mathrm{Var}\{\hat{\Phi}_{nn}(\omega)\} = \Gamma^2_{nn}(\omega) \tag{3.9}$$

Equation (3.9) states the intuitively obvious result, that the variance of the noise spectrum estimator does *not* reduce with increasing number of ensemble elements.

Substituting (2.11) into (3.8) the relative variance of the estimated signal spectrum becomes

$$\frac{\mathrm{Var}\{\hat{\Phi}_{ss}(\omega)\}}{\Gamma^2_{ss}(\omega)} = \frac{2}{(N\Theta(\omega))^2} + \frac{2}{N\Theta(\omega)} \tag{3.10}$$

Comparing (3.9) with (2.8) and (3.10) with (2.10) it becomes clear that the large savings in computation time must be paid for with an increase in variance. This concerns especially the noise spectrum estimator, which even becomes *inconsistent*, while the variance of the signal spectrum increases only slightly (Figure 3.1, the line R=1).

It should be noted that the same techniques for variance reduction as mentioned in Sect. 2 can be applied here as well.

4. Generalization of the Estimation Methods

By comparison of the two previous methods for separating the spectra of signal and noise it is obvious that, in a way, they represent opposite approaches. While with the method described in Sect. 2 the variance is minimized (at the cost of a large computational effort), with the method described in Sect. 3 the computational effort is minimized, but at the cost of a large variance. It seems that in practice neither of these extremes is preferable and therefore the question arises whether it is possible to find a compromise between both methods. It will be shown that by introducing the concept of *subensemble averaging* such a compromise is indeed possible. Moreover, it is shown that the methods described previously can be conceived of as special cases of two more general methods, which are very similar, both in their appearance as in their statistical properties.

In the following two subsections these generalized methods, based on alternate subensemble averaging and normal subensemble averaging respectively, will be considered and their merits discussed in the third subsection.

4.1 Ensemble and Alternate Subensemble Averaging

The basic idea of this method is illustrated in Figure 3.2. Instead of using the full ensemble for calculation of the alternate average and its correspon-

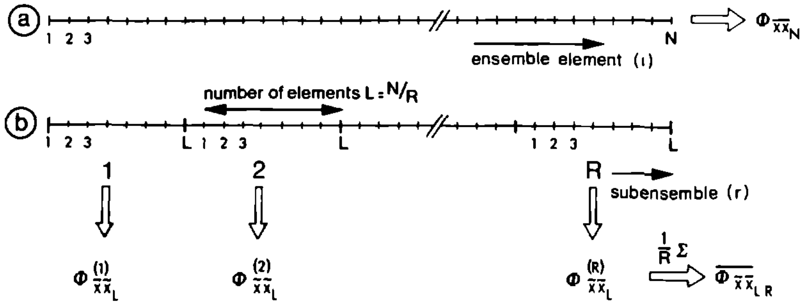

Figure 3.2a and b
Schematic representation of the method for separating signal and noise spectra as proposed in this paper. a: *Computation of the spectrum of the ensemble average;* b: *the same ensemble is divided into R subensembles, each of size L=N/R. In each subensemble the alternate average and the corresponding spectrum are computed. The resulting R spectra are averaged in order to reduce the variance. From* a *and* b *the underlying spectra of the "real" signal and the noise may be separated by simple mathematics*

ding spectrum $\Phi_{\tilde{x}\tilde{x}_N}$, we split up the ensemble into R subensembles, each of size L=N/R (L even). In each subensemble we compute an *alternate* average and its corresponding spectrum $\Phi^{(r)}_{\tilde{x}\tilde{x}_L}(\omega)$ (r = 1,2,.....,R). Finally, we average these spectra to obtain

$$\overline{\Phi_{\tilde{x}\tilde{x}_{L,R}}}(\omega) = \frac{1}{R}\sum_{r=1}^{R} \Phi^{(r)}_{\tilde{x}\tilde{x}_L}(\omega) \tag{4.1}$$

where the suffices L,R indicate alternate ensemble averaging over L elements at a time, followed by averaging over the corresponding R spectra.

Since the alternate averaging is now carried out over L instead of N elements it follows with (3.3) and (3.4) that:

$$E\{\overline{\Phi_{\tilde{x}\tilde{x}_{L,R}}}(\omega)\} = \frac{1}{R}\sum_{r=1}^{R} E\{\Phi^{(r)}_{\tilde{x}\tilde{x}_L}(\omega)\} = \Gamma_{\tilde{x}\tilde{x}_L}(\omega) = \frac{1}{L}\Gamma_{nn}(\omega) \tag{4.2}$$

so that the expectation of the estimators for signal and noise spectrum [cf. (3.5) and (3.6)] is transformed into

$$E\{\hat{\Phi}_{ss}(\omega)\} = \Gamma_{\overline{xx}_N}(\omega) - \frac{1}{R}\Gamma_{\tilde{x}\tilde{x}_L}(\omega) \tag{4.3}$$

and

$$E\{\hat{\Phi}_{nn}(\omega)\} = L\ \Gamma_{\tilde{x}\tilde{x}_L}(\omega) \tag{4.4}$$

The variance of the estimators can be determined in an analogous way as before. In Appendix 1 it is proven that $\mathrm{Cov}\{\Phi_{\overline{xx}_N}(\omega),\ \overline{\Phi_{\tilde{x}\tilde{x}_{L,R}}}(\omega)\}$ equals zero. Thus it follows with (A 10) and (A 19):

$$\mathrm{Var}\{\hat{\Phi}_{ss}(\omega)\} = \frac{\Gamma^2_{nn}(\omega)}{N^2}\ (1+\frac{1}{R}) + \frac{2}{N}\ \Gamma_{nn}(\omega)\Gamma_{ss}(\omega) \tag{4.5}$$

and

$$\mathrm{Var}\{\hat{\Phi}_{nn}(\omega)\} = \frac{1}{R}\ \Gamma^2_{nn}(\omega) \tag{4.6}$$

or, in terms of the relative variance, by substitution of (2.11):

$$\frac{\mathrm{Var}\{\hat{\Phi}_{ss}(\omega)\}}{\Gamma^2_{ss}(\omega)} = \frac{1+\frac{1}{R}}{(N\Theta(\omega))^2} + \frac{2}{N\Theta(\omega)} \tag{4.7}$$

and

$$\frac{\mathrm{Var}\{\hat{\Phi}_{nn}(\omega)\}}{\Gamma^2_{nn}(\omega)} = \frac{1}{R} \tag{4.8}$$

Note that by putting R=1 (L=N) we obtain the estimation procedure of Sect. 3. The other extreme is to put R=N/2 (L=2). In this case the ensemble elements are subtracted in pairs (i.e. $x_1 - x_2$, $x_3 - x_4, \ldots\ldots, x_{N-1} - x_N$) and their corresponding power density spectra averaged, which results in the minimum variance that can be achieved with this method as can be seen from (4.7) and (4.8). From (4.8) it appears that the present noise spectrum estimator is consistent. Its relative variance depends *only* on the number of subensembles R and is *independent of the ensemble size N.*

The influence of the number of subensembles R on the variance of the signal spectrum estimator is shown in Figure 3.1, as a function of the SNPDR of the ensemble average NΘ. From this figure, as well as from (4.7), it becomes obvious that enlargement of R beyond approximately 4-8 has a minor effect on the reduction of this variance. It should be noted that this holds true *independent of the ensemble size N,* although with increasing ensemble size the SNPDR of the average becomes larger, which in turn influences the relative weight of the first term appearing on the right side of (4.7).

4.2 Ensemble and Subensemble Averaging

Analogous to the method of the previous subsection we split up the ensemble into R subensembles of size L=N/R (in this case L need *not* be even). In each subensemble, however, we now compute a *normal* average and its corresponding spectrum $\Phi^{(r)}_{\overline{xx}_L}(\omega)$ $(r = 1,2,\ldots\ldots,R)$.

These spectra again are averaged, so that we obtain

$$\overline{\Phi_{\overline{xx}_{L,R}}}(\omega) = \frac{1}{R}\sum_{r=1}^{R}\Phi^{(r)}_{\overline{xx}_L}(\omega) \tag{4.9}$$

With (2.1) it follows that by replacing L instead of N:

$$E\{\Phi^{(r)}_{\overline{xx}_L}(\omega)\} = \Gamma_{\overline{xx}_L}(\omega) = \Gamma_{ss}(\omega) + \frac{1}{L}\Gamma_{nn}(\omega) \tag{4.10}$$

Taking expectations in (4.9) and substitution of (4.10) finally leads to

$$E\{\overline{\Phi_{\overline{xx}_{L,R}}}(\omega)\} = \Gamma_{ss}(\omega) + \frac{1}{L}\Gamma_{nn}(\omega) \tag{4.11}$$

With (4.11) and using the spectrum of the normal ensemble average (2.1) the expectation of the estimators for signal and noise spectrum [cf. (2.3) and (2.4)] is transformed into

$$E\{\hat{\Phi}_{ss}(\omega)\} = \frac{NL}{N-L}\{\frac{1}{L}\Gamma_{\overline{xx}_N}(\omega) - \frac{1}{N}\Gamma_{\overline{xx}_L}(\omega)\} \tag{4.12}$$

and

$$E\{\hat{\Phi}_{nn}(\omega)\} = \frac{NL}{N-L}\{\Gamma_{\overline{xx}_L}(\omega) - \Gamma_{\overline{xx}_N}(\omega)\} \tag{4.13}$$

The variance of these estimators is determined by using the results of Appendix 1. With (A 9), (A 10), and (A 12) it follows:

$$\mathrm{Var}\{\hat{\Phi}_{ss}(\omega)\} = \frac{\Gamma^2_{nn}(\omega)}{N^2(1-\frac{1}{R})} + \frac{2}{N}\Gamma_{nn}(\omega)\Gamma_{ss}(\omega) \tag{4.14}$$

and

$$\mathrm{Var}\{\hat{\Phi}_{nn}(\omega)\} = \frac{\Gamma^2_{nn}(\omega)}{R-1} \tag{4.15}$$

It is easily verified that by putting L=1 (R=N) we obtain the estimation procedure of Sect. 2. Note also that the other extreme is to put L=N/2 (R=2). [The case of L=N (R=1) is a non-existent method, since this implies that twice the same spectrum, i.e. $\Phi_{\overline{xx}_N}(\omega)$, is determined].

	Ensemble and subensemble averaging (Sect. 4.2)	Ensemble and alternate subensemble averaging (Sect. 4.1)
$E\{\hat\Phi_{ss}(\omega)\}$	$\frac{NL}{N-L}\left\{\frac{1}{L}\Gamma_{\overline{xx}_s}(\omega) - \frac{1}{N}\Gamma_{\overline{xx}_r}(\omega)\right\}$	$\Gamma_{\overline{xx}_s}(\omega) - \frac{1}{R}\Gamma_{\tilde{x}\tilde{x}_r}(\omega)$
$E\{\hat\Phi_{nn}(\omega)\}$	$\frac{NL}{N-L}\{\Gamma_{\overline{xx}_r}(\omega) - \Gamma_{\overline{xx}_s}(\omega)\}$	$L\Gamma_{\tilde{x}\tilde{x}_r}(\omega)$
$\frac{\mathrm{Var}\{\hat\Phi_{ss}(\omega)\}}{\Gamma^2_{ss}(\omega)}$	$\left(\frac{1}{1-\frac{1}{R}}\right)\frac{1}{(N\Theta(\omega))^2} + \frac{2}{N\Theta(\omega)}$	$\left(1+\frac{1}{R}\right)\frac{1}{(N\Theta(\omega))^2} + \frac{2}{N\Theta(\omega)}$
$\frac{\mathrm{Var}\{\hat\Phi_{nn}(\omega)\}}{\Gamma^2_{nn}(\omega)}$	$\frac{1}{R-1}$	$\frac{1}{R}$
spectral computations needed	1 spectrum of average R spectra of subensemble averages	1 spectrum of average R spectra of subensemble averages
Special cases	$L=1$ $R=N$ (Sect. 2)	$L=N$ $R=1$ (Sect. 3)
$E\{\hat\Phi_{ss}(\omega)\}$	$\frac{N}{N-1}\left\{\Gamma_{\overline{xx}_s}(\omega) - \frac{1}{N}\Gamma_{xx}(\omega)\right\}$	$\Gamma_{\overline{xx}_s}(\omega) - \Gamma_{\tilde{x}\tilde{x}_s}(\omega)$
$E\{\hat\Phi_{nn}(\omega)\}$	$\frac{N}{N-1}\{\Gamma_{xx}(\omega) - \Gamma_{\overline{xx}_s}(\omega)\}$	$N\Gamma_{\tilde{x}\tilde{x}_s}(\omega)$
$\frac{\mathrm{Var}\{\hat\Phi_{ss}(\omega)\}}{\Gamma^2_{ss}(\omega)}$	$\left(\frac{1}{1-\frac{1}{N}}\right)\frac{1}{(N\Theta(\omega))^2} + \frac{2}{N\Theta(\omega)}$	$\frac{2}{(N\Theta(\omega))^2} + \frac{2}{N\Theta(\omega)}$
$\frac{\mathrm{Var}\{\hat\Phi_{nn}(\omega)\}}{\Gamma^2_{nn}(\omega)}$	$\frac{1}{N-1}$	1
spectral computations needed	1 spectrum of average N spectra of ensemble elements	1 spectrum of average 1 spectrum of alternate average

Table 3.1 Summary of the methods for separating signal and noise spectra, discussed in this paper

4.3 Discussion

A synopsis of the previous findings is presented in Table 3.1. This table summarizes the generalized methods of the previous subsections, along with their "special cases", corresponding to the methods as discussed in Sects. 2 and 3. Obviously, both generalized procedures require the same amount of spectral calculations and the only difference is the normal as opposed to the alternate averaging within subensembles. Also with respect to the variance of the signal and noise spectrum estimators both procedures are virtually equal, the method using alternate subensemble averaging being only slightly better. This is, however, not the reason why the latter method is preferable, nor is the simple appearance of the expressions of the signal and noise spectrum estima-

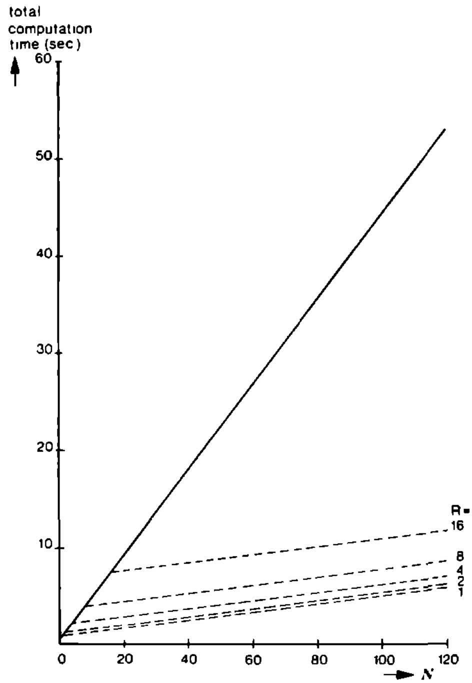

Figure 3.3

Comparison of the computation time needed for the separation of signal and noise spectra vs. the number of ensemble elements N. Solid line: *method of ensemble and ensemble spectra averaging;* broken lines: *newly proposed method, i.e. ensemble and alternate subensemble averaging, as a function of the number of subensembles R (for further details see text)*

tors (cf. Table 3.1). A factor of more importance is that in the practical application of averaging techniques one rarely deals with a strictly homogeneous ensemble. A large number of inhomogeneities is caused by trends, slow amplitude modulations of the signal or a slowly changing character of the background, i.e. in general by variations with a duration that is significantly longer than the duration of an ensemble element. The principle advantage then of alternate averaging is that due to the alternate addition and subtraction of *successive* ensemble elements such inhomogeneities tend to cancel out, which in turn implies that the corresponding estimator becomes less biased.

As far as the choice of the number of subensembles is concerned it can be concluded in view of the discussion in Sect. 4.1, that not much can be gained by choosing the number R above a certain limit. Even for small values of R (in the order of 4-8) one can come close to the (minimum) variance that should result when using the conventional method of Sect. 2, yet this is achieved with much less computational effort.

The estimation method as proposed in the foregoing requires R+1 calculations of power density spectra as opposed to N+1 calculations for the conventional case. This implies that, particularly when N is large, the savings are considerable. As a final example this is illustrated in Figure 3.3, where the total computation time needed for the separation of the spectra of signal and noise has been determined as a function of the number of ensemble elements N. In this case the ensemble was read from disk with 256 samples per element; computations were performed on a DEC PDP 11-34 computer with floating point processor using standard FFT routines.

5. Application to *a posteriori* "Wiener" Filtering

5.1 Introduction

Briefly, the idea behind *a posteriori* filtering is to improve the estimation of the "real" signal, as obtained after ensemble averaging, still further by a linear filtering procedure.

The appropriate filter transfer function is computed from the estimates of the underlying spectra of signal and noise (Walter, 1969; Doyle, 1975):

$$\hat{H}(\omega) = \frac{\hat{\Phi}_{ss}(\omega)}{\hat{\Phi}_{ss}(\omega) + \frac{1}{N}\hat{\Phi}_{nn}(\omega)} = \frac{\hat{\Phi}_{ss}(\omega)}{\overline{\Phi}_{xx_N}(\omega)} \tag{5.1}$$

De Weerd and Martens (1978) have shown that with decreasing signal-to-noise power density ratio (SNPDR) the transfer function according to (5.1) suffers from a progressively larger bias and variance due to the variability in the spectra from which this function is obtained. The authors also showed, however, that this bias and variance can be effectively reduced by spectral smoothing.

On the other hand smoothing, in general, causes a bias in the estimated spectra which in turn may result in an additional bias in the filter transfer function.

In order to obtain more theoretical insight into the consequences of smoothing, the first two moments of the filter transfer function have been approximated using an expansion in a Taylor series (Appendix 2). From Appendix 2 it follows that for a thorough evaluation it is necessary to have detailed knowledge of the shape of the underlying spectra of signal and noise, a knowledge which is lacking in practice. Therefore, as a simplification, we will assume that the underlying spectra are flat which implies that the smoothed spectral estimates are unbiased.

A problem of more fundamental importance is the fact that, although the former approximation may provide us with a certain insight into the statistical behaviour, such an approximation is meaningful only in the case where the joint probability density function of both spectra, as they appear in (5.1), is sufficiently concentrated near its center of gravity. Especially at low SNPDR values and/or small amounts of smoothing this seems most unlikely.

Therefore, the first two moments of the filter transfer function have also been estimated by a simulation, assuming again flat spectra for signal and noise. [For details of this simulation refer to de Weerd and Martens (1978)].

The computation of the transfer function by means of the spectral estimation method of Sect. 2 has been extensively discussed in the recent literature (cf. Sect. 1).

Therefore, in what follows, this method (Sect. 5.2) is compared with the newly proposed one (Sect. 5.3), using the above mentioned analysis procedures.

5.2 Ensemble and Ensemble Spectra Averaging

From (5.1) and (2.3) it follows, replacing the original spectra by their smoothed estimates, that

$$\hat{H}_1(\omega) = \frac{N}{N-1}\left(1 - \frac{1}{N}\,\frac{\overline{\Phi^s_{xx_N}}(\omega)}{\Phi^s_{\overline{xx}_N}(\omega)}\right) \tag{5.2}$$

As stated before we assume that the theoretical signal and noise spectra are flat. Furthermore we assume that smoothing is performed by means of a Tukey window with relative truncation point M/T (cf. Jenkins and Watts, 1968). Then, by substitution of $a = N/(N-1)$; $b = 1/(N-1)$; $\Phi_1 = \overline{\Phi_{xx_N}}$; $\Phi_2 = \Phi_{\overline{xx}_N}$ into (A 36) and using (A 10), (A 13) and (2.11) it follows:

$$E\{\hat{H}_1(\omega)\} \simeq \frac{N\Theta(\omega)}{N\Theta(\omega)+1} - \frac{3M}{4T}\,\frac{2N\Theta(\omega)+1}{(N\Theta(\omega)+1)^3} \tag{5.3}$$

The first term of (5.3) is the theoretical value of the transfer function; the second term represents a *negative bias* which depends on the SNPDR of the ensemble average and the degree of spectral smoothing.

In a similar way it follows from (A 37) after some straightforward calculations:

$$\mathrm{Var}\{\hat{H}_1(\omega)\} \cong \frac{3M}{4T}\frac{N^2(O(\omega)+1)^2 - N(N-1)}{(N-1)(NO(\omega)+1)^4} \tag{5.4}$$

This variance differs significantly from zero for NO lower than approximately 6. In this region, assuming N to be large, expression (5.4) may be further approximated by

$$\mathrm{Var}\{\hat{H}_1(\omega)\} \cong \frac{3M}{4T}\frac{2NO(\omega)+1}{(NO(\omega)+1)^4} \tag{5.5}$$

Results of the *simulation* study have been discussed in a previous article (de Weerd and Martens, 1978), and are briefly recapitulated in the next subsection.

5.3 Ensemble and Alternate Subensemble Averaging

The filter transfer function is now estimated using (5.1) and (4.3):

$$\hat{H}_2(\omega) = 1 - \frac{1}{R}\frac{\overline{\Phi^{s}_{\tilde{x}\tilde{x}}}_{L,R}(\omega)}{\Phi^{s}_{\overline{xx}_N}(\omega)} \tag{5.6}$$

Assuming again flat spectra and smoothing with a Tukey window it follows by substituting a=1; b=1/R; $\Phi_1 = \overline{\Phi_{\tilde{x}\tilde{x}}}_{L,R}$; $\Phi_2 = \Phi_{\overline{xx}_N}$ into (A 36) and by using the results of Appendix 1:

$$E\{\hat{H}_2(\omega)\} \cong \frac{NO(\omega)}{NO(\omega)+1} - \frac{3M}{4T}\frac{2NO(\omega)+1}{(NO(\omega)+1)^3} \tag{5.7}$$

and similarly

$$\mathrm{Var}\{\hat{H}_2(\omega)\} \cong \frac{3M}{4T}\left\{\frac{2NO(\omega)+1}{(NO(\omega)+1)^4} + \frac{1}{R}\frac{1}{(NO(\omega)+1)^2}\right\} \tag{5.8}$$

Comparing these moments with (5.3) and (5.5) and recalling that we assumed flat spectra the following conclusions may be drawn:

- both estimation methods have an identical expectation; they suffer from a negative bias which is *independent* of the number of subensembles
- as compared to the conventional method of Sect. 2 the newly proposed method adds a second term to the expression for the variance; this term reduces proportionally with the number of subensembles R.

These theoretical findings are confirmed by the simulation studies. The mean value of the transfer function obtained by the simulations is equal for

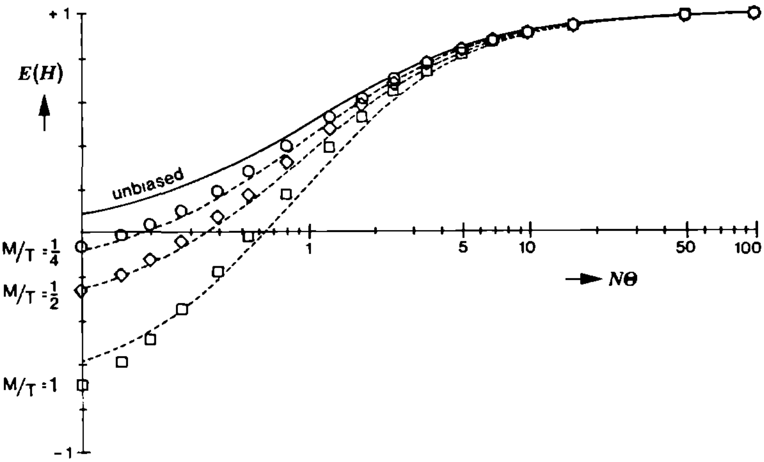

Figure 3.4

Approximated and estimated mean values of the a posteriori *filter transfer function vs. the SNPDR of the ensemble average NΘ (N=100) assuming flat spectra of signal and noise. Smoothing is performed using a Tukey window with relative truncation point M/T.* Symbols: *values obtained by simulation.* Broken lines: *approximated theoretical mean values using an expansion in a Taylor series. From the theoretical as well as from the simulation point of view, both spectral estimation methods as discussed in Sect. 5 yield identical results. For comparison, the unbiased value of the transfer function is also shown* (solid line)

both spectral estimation methods. Results are shown in Figure 3.4 (symbols) for various relative truncation points M/T of the Tukey window, vs. NΘ, the SNPDR of the ensemble average. Also shown are the paths of the approximated theoretical curves (broken lines), using (5.3) and the theoretical *unbiased* curve (solid line). Obviously, the agreement with the simulated values is satisfactory.

In Figure 3.5 the variance of the transfer function based on the two different spectral estimation methods is compared, again for various relative truncation points of the Tukey window. The solid lines show the variance, obtained by simulation, using the conventional spectral estimation method (data from de Weerd and Martens, 1978). The broken lines represent the variance, again obtained by simulation, using the new spectral estimation method with

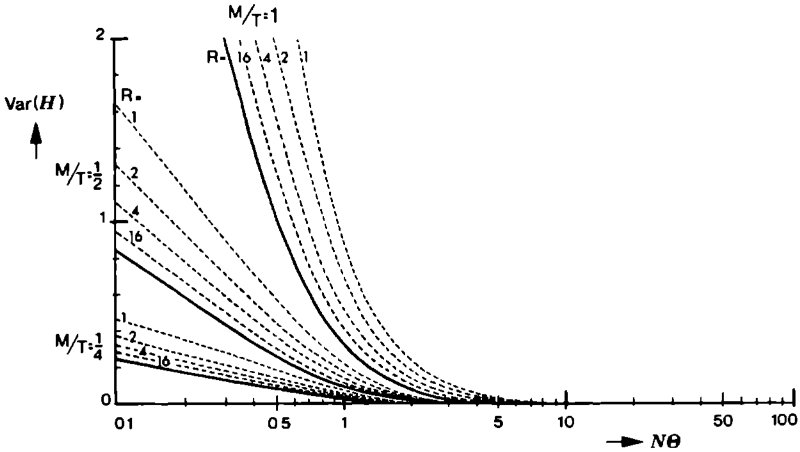

Figure 3.5

Estimated variance of the a posteriori *filter transfer function vs. the SNPDR of the ensemble average NΘ(N=100). All curves are obtained by simulation assuming flat spectra of signal and noise. Smoothing is performed with a Tukey window with relative truncation point M/T.* Solid lines: *estimation of transfer function using the method of ensemble and ensemble spectra averaging (Sect. 5.2).* Broken lines: *estimation of transfer function using the method of ensemble and alternate subensemble averaging (Sect. 5.3); the number of subensembles is given by R*

the number of subensembles R as parameter. From this figure we can draw a similar conclusion to that drawn previously, i.e. that the number of subensembles R should not be arbitrarily large. It should be noted that this conclusion also holds true when the spectra of signal and noise are not flat, but have an arbitrary shape. Figure 3.5 also suggests that the amount of smoothing (or the relative truncation points M/T of the spectral window) is of much more importance to the reduction of variance than is the number of subensembles. However, when the simplifying assumption that the theoretical spectra of signal and noise were flat is *not* valid this conclusion may be quite different.

6. Conclusion

The methods for separation of signal and noise spectra as discussed in the previous sections are applicable when dealing with an ensemble in which each element is conceptually composed of an invariant signal and an additive stationary noise component, uncorrelated with this signal.

The estimation of the signal spectrum involves a subtraction of two spectra, the *first* always being the spectrum of the ensemble average [1].

Using a more common separation method (Sect. 2), the additionally computed spectrum, which in the remaining discussion we will refer to as the *second* spectrum, is an average of spectra of N individual ensemble elements. It has been shown that with this method the signal spectrum estimator has a minimum variance, but this is achieved at a large computational effort.

The generalized methods for separation of the signal and noise spectra as presented in this paper are based on the notion that a large increase in variance of the second spectrum can be allowed before this results in a significant increase in the variance of the ultimate signal spectrum estimator. Therefore, in these methods, the second spectrum consists of an average of only R spectra (usually $R \ll N$), where each spectrum is based on (alternate) averaged subensembles (Sect. 4).

This implies that large savings in computation time are possible since the necessary power density spectra computations are approximately reduced by a factor N/R as compared to the conventional method.

This fact in turn allows multichannel application or extension of these methods to the estimation of *time-varying* signal and noise spectra, a procedure which follows the same lines as set forth in this paper. Another advantage of these methods is their flexibility, because they permit, by choosing the number of subensembles R, a weighing of the admissible relative increase in variance against the available computation time.

Of the generalized methods one seems particularly useful in practice, i.e. the method of ensemble and *alternate* subensemble averaging (Sect. 4.1). Due to the alternate addition and subtraction of subsequent ensemble elements this method is less susceptible to inhomogeneities due to slow variations within the ensemble.

Finally, irrespective of the particular method that is employed for the estimation of the signal spectrum, one should be aware of the large variance in the estimate at low signal-to-noise power density ratios (cf. Figure 3.1). If some bias and loss of frequency resolution are admissible, smoothing provides an effective means of improving this estimate. Mutatis mutandis the same remarks apply to the estimation of the filter transfer function when using *a posteriori* "Wiener" filtering.

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1) It can be shown that methods for separation of signal and noise spectra not [illegible]ing use of the spectrum of the ensemble average lead to a larger variance.

Appendix 1

The average power density spectrum of the sub-ensemble averages [cf. (1.2) and (4.9)] is given by

$$\overline{\Phi_{\tilde{x}\tilde{x},L,R}}(\omega)=\frac{1}{R}\sum_{r=0}^{R-1}\left|\frac{1}{L}\sum_{i=rL+1}^{(r+1)L} S(\omega)+N_i(\omega)\right|^2 \tag{A1}$$

and the average power density spectrum of the *alternate* sub-ensemble averages [cf. (3.1) and (4.1)] by

$$\overline{\Phi_{\check{x}\check{x},L,R}}(\omega)=$$
$$=\frac{1}{R}\sum_{r=0}^{R-1}\left|\frac{1}{L}\sum_{i=rL+1}^{(r+1)L}(-1)^{i-1}(S(\omega)+N_i(\omega))\right|^2 \quad (L\text{ even}) \tag{A2}$$

From (A1) it follows with (4.11):

$$\operatorname{Var}\{\overline{\Phi_{\tilde{x}\tilde{x},L,R}}(\omega)\}=$$
$$=E\left\{\left(\frac{1}{R}\sum_{r=0}^{R-1}\left|\frac{1}{L}\sum_{i=rL+1}^{(r+1)L} S(\omega)+N_i(\omega)\right|^2\right)^2\right\}+$$
$$-\left(\Gamma_{ss}(\omega)+\frac{1}{L}\Gamma_{nn}(\omega)\right)^2 \tag{A3}$$

Since by substitution of $L=N$, $R=1$ in (A1) we obtain $\Phi_{\tilde{x}\tilde{x},N}(\omega)$, we also have

$$\operatorname{Cov}\{\Phi_{\tilde{x}\tilde{x},N}(\omega),\overline{\Phi_{\tilde{x}\tilde{x},L,R}}(\omega)\}=$$
$$=E\left\{\left|\frac{1}{N}\sum_{i=1}^{N} S(\omega)+N_i(\omega)\right|^2\cdot\right.$$
$$\left.\frac{1}{R}\sum_{r=0}^{R-1}\left|\frac{1}{L}\sum_{k=rL+1}^{(r+1)L} S(\omega)+N_k(\omega)\right|^2\right\}+$$
$$-\left(\Gamma_{ss}(\omega)+\frac{1}{N}\Gamma_{nn}(\omega)\right)\left(\Gamma_{ss}(\omega)+\frac{1}{L}\Gamma_{nn}(\omega)\right) \tag{A4}$$

From (A2) it follows with (4.2):

$$\operatorname{Var}\{\overline{\Phi_{\check{x}\check{x},L,R}}(\omega)\}=$$
$$=E\left\{\left(\frac{1}{R}\sum_{r=0}^{R-1}\left|\frac{1}{L}\sum_{i=rL+1}^{(r+1)L}(-1)^{i-1}(S(\omega)+N_i(\omega))\right|^2\right)^2\right\}+$$
$$-\frac{1}{L^2}\Gamma_{nn}^2(\omega) \tag{A5}$$

and

$$\operatorname{Cov}\{\Phi_{\tilde{x}\tilde{x},N}(\omega),\overline{\Phi_{\check{x}\check{x},L,R}}(\omega)\}=$$
$$=E\left\{\left|\frac{1}{N}\sum_{i=1}^{N} S(\omega)+N_i(\omega)\right|^2\right.$$
$$\left.\frac{1}{R}\sum_{r=0}^{R-1}\left|\frac{1}{L}\sum_{k=rL+1}^{(r+1)L}(-1)^{k-1}(S(\omega)+N_k(\omega))\right|^2\right\}+$$
$$-\frac{1}{L}\Gamma_{nn}(\omega)\left(\Gamma_{ss}(\omega)+\frac{1}{N}\Gamma_{nn}(\omega)\right) \tag{A6}$$

It will be shown that from these basic relations the moments of the other estimators may be derived easily.

In order to simplify the analysis, we define the function

$$\Pi(i,j,k,m)\stackrel{\Delta}{=}E\{(S+N_i)(S^*+N_j^*)(S+N_k)(S^*+N_m^*)\} \tag{A7}$$

where the argument ω has been omitted for reasons of brevity. The function Π is evaluated using the following diagram:

$E\{\ \}$	SS^*	S^*N_i	SN_j^*	$N_iN_j^*$
SS^*	Γ_{ss}^2	0	0	$\Gamma_{ss}\Gamma_{nn}(i-j)$ $0(i\neq j)$
S^*N_k	0	0	$\Gamma_{ss}\Gamma_{nn}(j-k)$ $0(j\neq k)$	0
SN_m^*	0	$\Gamma_{ss}\Gamma_{nn}(i-m)$ $0(i\neq m)$	0	0
$N_kN_m^*$	$\Gamma_{ss}\Gamma_{nn}(k-m)$ $0(k\neq m)$	0	0	$E\{N_iN_j^*N_kN_m^*\}$

The zero values in the diagram appear due to the fact that the respective products are complex numbers with random phase and intensity, thus having zero expectation.

For the same reason the noise term $E\{N_iN_j^*N_kN_m^*\}$ is only non zero if $i=j$ and $k=m$ or $i=m$ and $j=k$. In particular

$$E\{N_iN_j^*N_kN_m^*\}=\Gamma_{nn}^2 \quad \text{if} \quad (i=j,\ k=m \text{ and } i\neq k)$$
$$\text{or} \quad (i=m,\ j=k \text{ and } i\neq j)$$
$$=\operatorname{Var}\{\Phi_{nn}\}+E^2\{\Phi_{nn}\}=2\Gamma_{nn}^2 \quad (i=j=k=m)^*$$
$$=0 \text{ in all other cases.} \tag{A8}$$

When substituting the function (A7) into the expressions (A3)–(A6) the only problem that remains is to determine the number of times that a particular term appears in the summation.

It follows

$$\operatorname{Var}\{\overline{\Phi_{\tilde{x}\tilde{x},L,R}}\}=$$
$$=\frac{1}{R^2L^4}\sum_{p=0}^{R-1}\sum_{q=0}^{R-1}\sum_{i=pL+1}^{(p+1)L}\sum_{j=pL+1}^{(p+1)L}\sum_{k=qL+1}^{(q+1)L}\sum_{m=qL+1}^{(q+1)L}\Pi(i,j,k,m)+$$
$$-\left(\Gamma_{ss}+\frac{1}{L}\Gamma_{nn}\right)^2=$$
$$=\frac{1}{R^2L^4}(R^2L^4\Gamma_{ss}^2+(2R^2L^3+2RL^3)\Gamma_{nn}\Gamma_{ss}+$$
$$+(N(N-1)+2N+RL(L-1))\Gamma_{nn}^2)-\left(\Gamma_{ss}+\frac{1}{L}\Gamma_{nn}\right)^2=$$
$$=\frac{\Gamma_{nn}^2}{NL}+\frac{2}{N}\Gamma_{nn}\Gamma_{ss} \tag{A9}$$

which for the special case that $L=N$, $R=1$ reduces to

$$\operatorname{Var}\{\Phi_{\tilde{x}\tilde{x},N}\}=\frac{\Gamma_{nn}^2}{N^2}+\frac{2}{N}\Gamma_{nn}\Gamma_{ss} \tag{A10}$$

* See Jenkins and Watts (1968)

and for the case that $I=1$, $R=N$ to

$$\operatorname{Var}\{\overline{\Phi_{xx}}\}=\frac{I_{nn}^2}{N}+\frac{2}{N}I_{nn}I_{ss} \tag{A 11}$$

Similarly it follows for the covariance

$$\operatorname{Cov}\{\Phi_{\overline{xx}},\overline{\Phi_{\overline{xx},R}}\}=$$

$$=\frac{1}{N^2RL^2}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{r=0}^{R-1}\sum_{k=rL+1}^{(r+1)L}\sum_{m=rL+1}^{(r+1)L}\Pi(i,j,k,m)+$$

$$-\left(I_{ss}+\frac{1}{N}I_{nn}\right)\left(I_{ss}+\frac{1}{L}I_{nn}\right)=$$

$$=\frac{1}{N^3L}(N^3LI_{ss}^2+(N^3+3N^2L)I_{nn}I_{ss}+$$

$$+(N(N-1)+2N+RL(L-1))I_{nn}^2)+$$

$$-\left(I_{ss}+\frac{1}{N}I_{nn}\right)\left(I_{ss}+\frac{1}{L}I_{nn}\right)=\frac{I_{nn}^2}{N^2}+\frac{2}{N}\Gamma_{nn}I_{ss} \tag{A 12}$$

Since this expression is independent of L and R this also implies that

$$\operatorname{Cov}\{\Phi_{\overline{xx}},\overline{\Phi_{xx}}\}=\frac{I_{nn}^2}{N^2}+\frac{2}{N}\Gamma_{nn}I_{ss} \tag{A 13}$$

Analogously by substituting (A 7) into (A 5) we obtain

$$\operatorname{Var}\{\overline{\Phi_{xx,I,R}}\}=$$

$$=\frac{1}{L^4R^2}\sum_{p=0}^{R-1}\sum_{q=0}^{R-1}\sum_{i=pL+1}^{(p+1)L}\sum_{j=pL+1}^{(p+1)L}\sum_{k=qL+1}^{(q+1)L}\sum_{m=qL+1}^{(q+1)L}(-1)^{i+j+k+m-4}$$

$$\Pi(i,j,k,m)-\frac{1}{L^2}I_{nn}^2 \tag{A 14}$$

Due to the alternating sign and the fact that L is an *even* number all terms in this summation yield a net contribution of zero except for the noise term $F\{N_iN_j^*N_kN_m^*\}$.

According to (A 8) we can distinguish the following cases.

For $i=j$ and $k=m$ we have

$$\frac{1}{L^4R^2}\sum_{p=0}^{R-1}\sum_{q=0}^{R-1}\sum_{i=pL+1}^{(p+1)L}\sum_{k=qL+1}^{(q+1)L}F\{N_iN_i^*N_kN_k^*\} \tag{A 15}$$

which for $i\neq k$ and using the relation $N=RL$ reduces to

$$\frac{1}{L^4R^2}\sum_{i=1}^{N}\sum_{k=1}^{N}E\{N_iN_i^*N_kN_k^*\}=$$

$$=\frac{N(N-1)}{L^4R^2}\Gamma_{nn}^2 \quad (i\neq k) \tag{A 16}$$

while for $i=k$, which implies that $p=q$, it follows

$$\frac{1}{L^4R^2}\sum_{p=0}^{R-1}\sum_{i=pL+1}^{(p+1)L}E\{\Phi_{nn}^2\}=\frac{2N}{L^4R^2}I_{nn}^2 \quad (i=k) \tag{A17}$$

For $i=m$ and $j=k$ $(i\neq j)$ (which since i and m as well as j and k must lie in the same subensemble implies that $p=q$) (A 15) reduces to

$$\frac{1}{L^4R^2}\sum_{p=0}^{R-1}\sum_{i=pL+1}^{(p+1)L}\sum_{j=pL+1}^{(p+1)L}F\{N_iN_i^*N_jN_j^*\}=$$

$$=\frac{RL(L-1)}{L^4R^2}I_{nn}^2 \quad (i\neq j) \tag{A 18}$$

Substitution of (A 16), (A 18) into (A 14) finally yields

$$\operatorname{Var}\{\overline{\Phi_{xx,I,R}}\}=\frac{1}{NL}I_{nn}^2 \tag{A 19}$$

and for the special case $L=N$ (cf. Sect. 3)

$$\operatorname{Var}\{\Phi_{xx}\}=\frac{1}{N^2}I_{nn}^2 \tag{A 20}$$

At first sight the expression (A 19) for the variance seems to imply a contradictory result, since it suggests that the variance becomes smaller with *increasing* size L and thus *decreasing* number R of sub ensembles (we recall that $N=RL$). It should be noted, however, that when the size L increases the expected value of the spectrum $\overline{\Phi_{xx,I,R}}(\omega)$ becomes smaller [cf. (4 2)] so that the relative variance

$$\frac{\operatorname{Var}\{\overline{\Phi_{xx,I,R}}\}}{I_{xx,I}^2}=\frac{L}{N}=\frac{1}{R} \tag{A21}$$

becomes proportionally larger with decreasing number of sub-ensembles, as would be expected theoretically.

The covariance (A 6) follows from

$$\operatorname{Cov}\{\Phi_{\overline{xx}},\overline{\Phi_{xx,I,R}}\}=$$

$$=\frac{1}{N^3L}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{r=0}^{R-1}\sum_{k=rL+1}^{(r+1)L}\sum_{m=rL+1}^{(r+1)L}(-1)^{k+m-2}$$

$$\Pi(i,j,k,m)-\frac{1}{L}I_{nn}\left(I_{ss}+\frac{1}{N}\Gamma_{nn}\right) \tag{A22}$$

In this case the only two terms that have a total contribution which is non zero are

$E\{SS^*N_kN_m^*\}$ $(k=m)$

and

$E\{N_iN_j^*N_kN_m^*\}$ $((i=j,k=m)$ or $(i=m,j=k))$

The first term can be rewritten as

$$\frac{1}{N^3 L}\sum_{i=1}^{N}\sum_{j=1}^{N}\sum_{k=1}^{N} E\{SS^* N_k N_k^*\} = \frac{1}{L}\Gamma_{ss}\Gamma_{nn} \tag{A23}$$

and the second as the sum of

$$\frac{1}{N^3 L}\sum_{i=1}^{N}\sum_{k=1}^{N} E\{N_i N_i^* N_k N_k^*\} =$$
$$= \frac{N(N-1)+2N}{N^3 L}\Gamma_{nn}^2 \quad (i=j,\ k=m,\ i=k \text{ included}) \tag{A 24}$$

and

$$\frac{1}{N^3 L}\sum_{k=1}^{N}\sum_{m=1}^{N}(-1)^{k+m-2} E\{N_k N_k^* N_m N_m^*\} =$$
$$= -\frac{1}{N^2 L}\Gamma_{nn}^2 \quad (i=m, j=k, k \neq m) \tag{A 25}$$

The latter result is easily verified as follows

Since there are N terms ($k=m$) excluded from this summation ($k+m$ even) there remain N terms ($k+m$ odd) with a negative contribution, all other terms cancel each other out because of the alternating sign

Finally substitution of (A23) (A25) into (A22) results in a covariance term which equals zero for all values of L and N.

Appendix 2

Let the transfer function of the a posteriori estimated filter be given by the general expression

$$\hat{H}(\omega) = a - b\frac{\Phi_1^s(\omega)}{\Phi_2^s(\omega)} \tag{A 26}$$

where a and b are constants, and $\Phi_1^s(\omega)$ and $\Phi_2^s(\omega)$ represent smoothed power density spectra.

For these power density spectra we assume the following model (the argument ω is dropped again for reasons of brevity)

$$\Phi_i^s = \Gamma_i + B\{\Phi_i^s\} + \varepsilon\{\Phi_i^s\} \quad i=1,2 \tag{A 27}$$

where Γ_i represents the theoretical power density spectrum, $B\{\Phi_i^s\}$ the bias of the smoothed spectrum and $\varepsilon\{\Phi_i^s\}$ a random spectral error with zero mean Note that according to the above definitions.

$$B\{\Phi_i^s\} = E\{\Phi_i^s\} - \Gamma_i \quad i=1,2 \tag{A 28}$$

$$E\{\varepsilon^2(\Phi_i^s)\} = \mathrm{Var}\{\Phi_i^s\} \quad i=1,2 \tag{A 29}$$

and

$$E\{\varepsilon(\Phi_1^s)\varepsilon(\Phi_2^s)\} = \mathrm{Cov}\{\Phi_1^s, \Phi_2^s\} \tag{A 30}$$

Expanding (A 26) into a Taylor series about (Γ_1, Γ_2) we obtain

$$\hat{H} = a - b\frac{\Gamma_1}{\Gamma_2} - \frac{b}{\Gamma_2}(B\{\Phi_1^s\} + \varepsilon\{\Phi_1^s\}) +$$
$$+ \frac{b\Gamma_1}{\Gamma_2^2}(B\{\Phi_2^s\} + \varepsilon\{\Phi_2^s\}) +$$
$$+ \frac{b}{\Gamma_2^2}(B\{\Phi_1^s\} + \varepsilon\{\Phi_1^s\})(B\{\Phi_2^s\} + \varepsilon\{\Phi_2^s\}) +$$
$$- \frac{b\Gamma_1}{\Gamma_2^3}(B\{\Phi_2^s\} + \varepsilon\{\Phi_2^s\})^2 + \tag{A 31}$$

Taking expectations and retaining up to second order moments it follows after some manipulations:

$$E\{\hat{H}\} \cong a - b\frac{\Gamma_1}{\Gamma_2} + \frac{b}{\Gamma_2^3}((\Gamma_2 - B\{\Phi_2^s\})(\Gamma_1 B\{\Phi_2^s\} +$$
$$- \Gamma_2 B\{\Phi_1^s\}) +$$
$$- \Gamma_1 \mathrm{Var}\{\Phi_2^s\} + \Gamma_2 \mathrm{Cov}\{\Phi_1^s, \Phi_2^s\}) \tag{A 32}$$

From (A 32) and (A 31) it follows for the approximate variance, again within second order moments:

$$\mathrm{Var}\{\hat{H}\} \cong \frac{b^2}{\Gamma_2^4}\Big((\Gamma_2 - B\{\Phi_2^s\})^2 \mathrm{Var}\{\Phi_1^s\} +$$
$$+ \Big(\Gamma_1 + B\{\Phi_1^s\} - \frac{2\Gamma_1}{\Gamma_2}B\{\Phi_2^s\}\Big)^2 \mathrm{Var}\{\Phi_2^s\} +$$
$$- 2(\Gamma_2 - B\{\Phi_2^s\})\Big(\Gamma_1 + B\{\Phi_1^s\} +$$
$$- \frac{2\Gamma_1}{\Gamma_2}B\{\Phi_2^s\}\Big)\mathrm{Cov}\{\Phi_1^s, \Phi_2^s\}\Big) \tag{A 33}$$

Expressions (A 32) and (A 33) may be further evaluated, if the smoothing window is known, e g. for a Tukey window we have (Jenkins and Watts, 1968).

$$B\{\Phi_i^s\} \cong \frac{0\,063}{M^2}\Gamma_i'' \tag{A 34}$$

and

$$\mathrm{Var}\{\Phi_i^s\} \cong \frac{3M}{4T}\mathrm{Var}\{\Phi_i\} \tag{A 35}$$

where Γ'' is the second derivative of the spectrum Γ, M represents the truncation point of the window and T the observation time over which the spectrum is calculated The quotient M T thus can be interpreted as the *relative* truncation point of the window.

From (A 34) it follows that evaluation of the spectral bias, and thus of the first two moments of the transfer function, requires detailed knowledge of the shape of the underlying spectra

Assuming, as a simplification, that the underlying spectra are flat the bias terms become zero In that case, using a Tukey window, the first two moments of the transfer function reduce to:

$$E\{\hat{H}\} \cong a - b\frac{\Gamma_1}{\Gamma_2} - \frac{3M}{4T}\frac{b}{\Gamma_2^3}$$
$$\cdot(\Gamma_1 \mathrm{Var}\{\Phi_2\} - \Gamma_2 \mathrm{Cov}\{\Phi_1, \Phi_2\}) \tag{A 36}$$

and

$$\mathrm{Var}\{\hat{H}\} \cong \frac{3M}{4T}\frac{b^2}{\Gamma_2^4}(\Gamma_2^2 \mathrm{Var}\{\Phi_1\} + \Gamma_1^2 \mathrm{Var}\{\Phi_2\} +$$
$$- 2\Gamma_1\Gamma_2 \mathrm{Cov}\{\Phi_1, \Phi_2\}) \tag{A 37}$$

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Chapter 4

Facts and Fancies about *a posteriori* "Wiener" Filtering

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Abstract

A posteriori "Wiener" filtering is an estimation technique that has gained renewed interest in recent years, particularly in evoked potential studies. However, several misconceptions appear to exist with respect to the theory and its applicability. The purpose of this paper is to clarify these aspects and to explain the method in a comprehensive manner. Finally, the use of a more powerful time-varying filtering technique is recommended.

1. Introduction

In a recent paper Carlton and Katz (1980) examine the effectiveness of "Wiener" filtering in improving the estimation of averaged evoked potentials. Based on some simulations, the authors conclude that no improvement can be expected in the mean-square error sense.

It appears from this study, however, that several misconceptions exist, both with respect to the theory and its applicability. A discussion of these aspects is, we believe, of more general importance, particularly in view of the renewed interest in "Wiener" filtering as reflected in the substantial number of papers that have appeared in the last five years (Doyle, 1975; Albrecht and Radil-Weiss, 1976; Hartwell and Erwin, 1976; Ungan and Basar, 1976; Albrecht et al, 1977; Strackee and Cerri, 1977; Kaveh et al, 1978; Naitoh and Sunderman, 1978; de Weerd and Martens, 1978; Carlton and Katz, 1979, 1980; Kearney, 1979; Nagelkerke and Strackee, 1979; de Weerd et al, 1979).

The main purpose of this paper is to explain some of the background and merits of *a posteriori* "Wiener" filtering (*a.p.w.f.*) at a conceptual, rather than at a rigorous mathematical level. It is also argued, that the effectiveness of a.p.w.f. in evoked potential estimation cannot be assessed in general. The effectiveness depends on many factors, such as the transiency of the evoked potential waveform, the amount of spectral overlap with the background activity and the actual "signal-to-noise" ratio of the ensemble average to which a.p.w.f. is applied. In practice these factors vary substantially - e.g., for the different sensory modalities - so that general statements concerning the applicability of the a.p.w.f. method do not seem fully warranted. However, a more powerful estimation technique for evoked potentials does exist, as will briefly be described.

2. What is *a posteriori* "Wiener" Filtering?

Wiener's original theory (Wiener, 1964) solves the problem of optimal signal estimation in the presence of additive noise, assuming that both the signal and the noise are stochastic and stationary processes with known power density spectra (see e.g. Papoulis, 1965, for a comprehensive formulation). In mathematical terms we have

$$x(t) = s(t) + n(t) \tag{2.1}$$

where $s(t)$ represents the (stochastic) signal and $n(t)$ the noise. If signal and noise have zero mean and are uncorrelated, the optimal filter according to Wiener is given by

$$H(f) = \frac{\Gamma_{ss}(f)}{\Gamma_{ss}(f) + \Gamma_{nn}(f)} \tag{2.2}$$

where $H(f)$ is the filter transfer function and $\Gamma_{ss}(f)$ and $\Gamma_{nn}(f)$ the *(a priori)* known power density spectra of the signal and the noise. This filter applied to $x(t)$ leads to an optimal estimate of the signal $s(t)$ in the mean-square error sense

$$\hat{s}(t) = F^{-1}\{H(f)X(f)\} \tag{2.3}$$

where $\hat{s}(t)$ represents the *optimally filtered* signal; $X(f)$ the Fourier transform of $x(t)$ and F^{-1} the inverse Fourier transform. It should be noted that $H(f)$ is a purely real and thus zero-phase function. This implies that Wiener filtering does not introduce any phase distortion.

The Wiener filter in fact weighs the spectral components in $X(f)$ according to the signal-to-noise power density ratio at individual frequencies (Figure 4.1). In regions where there is no signal power the spectral components are entirely suppressed; if there is no noise power, the components are entirely passed. However, in regions where signal and noise show spectral overlap, the filter not only affects the noise components in $X(f)$, but the signal components as well. The Wiener filter is therefore, in general, a *biased* estimator;

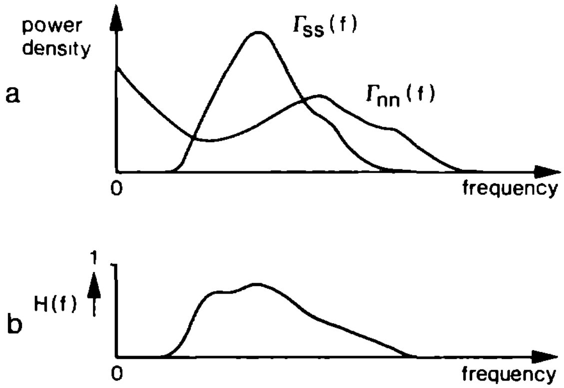

Figure 4.1a and b
The Wiener filter is a linear time-invariant filter for the optimal estimation of a stochastic and stationary signal perturbed with additive stationary noise. a: *Calculation of this filter requires* a priori *knowledge of the power density spectra of the signal* (Γ_{ss}) *and the noise* (Γ_{nn}). b: *The transfer function H(f) is essentially based on the ratio of these spectra (cf. 2.2)*

it reduces the variance in x(t) but at the cost of an increased bias, that is, a *systematic* reduction of the amplitude of the signal components. By minimizing the mean-square error (MSE) the filter attains a compromise between these two factors, as can be seen from the following simple relation (see e.g. Jenkins and Watts, 1968)

$$\text{MSE} = \text{variance} + (\text{bias})^2 \tag{2.4}$$

(see also Figure 4.4). From the above it may be clear that the smaller the spectral overlap between signal and noise, the more effective the Wiener filter is.

Now the central idea of a.p.w.f., which is due to Walter (1969), is that an *averaged* signal that is still contaminated with noise could be improved in the same way as shown above, however, based on *a posteriori estimated* power density spectra of the signal and the noise. One possible way (Walter, 1969) to estimate these spectra from the ensemble making up the average, is by computing the power density spectrum of the ensemble average denoted by $\Phi_{\overline{xx}}$ and the average of the power density spectra of individual ensemble elements, denoted by $\overline{\Phi_{xx}}$ (Figure 4.2). However, two problems arise.

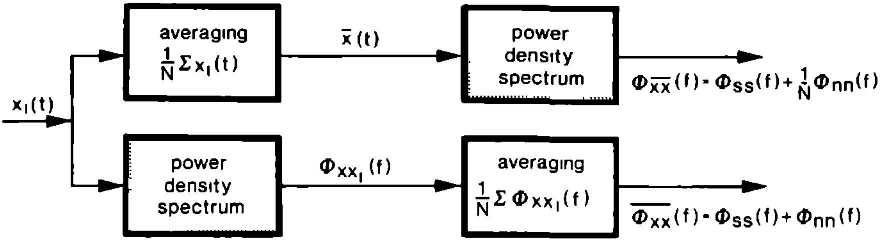

Figure 4.2
In practice the power density spectra of the signal and the noise are unknown. But when Wiener filtering is applied to an averaged waveform, these spectra can be estimated *from the ensemble making up this average. One possible way to do this is to compute the power density spectrum of the ensemble average $\Phi_{\bar{x}\bar{x}}$ (top) and the average of the power density spectra of individual ensemble elements $\overline{\Phi_{xx}}$ (bottom). From these spectra the estimated underlying power density spectra of the signal (Φ_{ss}) and the noise (Φ_{nn}) can be calculated by simple mathematics. Note that this procedure only makes sense if the signal is assumed to be* deterministic, *rather than stochastic*

In the first place, Walter's method, as well as generalized methods recently introduced for estimation of signal and noise spectra (de Weerd et al, 1979), assumes that the signal-to-noise power ratio increases directly proportional to the number of averaged ensemble elements. This only holds true if the signal is a fixed or *deterministic* signal, so that the average waveform $\bar{x}(\tau)$ is given by

$$\bar{x}(\tau) = s(\tau) + \frac{1}{N}\sum_{i=1}^{N} n_i(\tau) \qquad 0\leq\tau\leq T \tag{2.5}$$

where $n_i(\tau)$ is the i^{th} ensemble element of the noise process and T the duration of the evoked potential. However, such an assumption is clearly at variance with the assumed *stochastic* and *stationary* signal in Wiener's theory. Secondly, the power density spectra appearing in (2.2) become *estimated*, rather than *known* quantities, which in turn implies that the transfer function itself becomes an estimate.

The first problem concerning the deterministic nature of the signal has not been investigated in depth, but intuitively it appears that as long as the signal has a power distribution which is time-invariant or, in other words, has a short-time spectrum which is independent of time, the a.p.w.f. approach may remain valid. In that case the power distribution can fully be described by power density spectra, where the presence of infinite wavetrains is implicitly assumed. Examples of such signals can be found in the class of *sinusoidal* or similar periodic signals. On the contrary, the power distribution of *transient* signals can only be described adequately by *time-varying spectra*. For this class of signals the method of a.p.w.f. is, just like the original Wiener theory, obviously not suitable. We will come back to this point in a subsequent section.

The second problem of the transfer function being itself an estimate, has been extensively investigated by Strackee and Cerri (1977) and de Weerd and Martens (1978). These investigations have led to the conclusion that the *estimated* transfer function suffers from a large variance and negative bias at low signal-to-noise power density ratios and, curiously, may even take negative values despite the fact that numerator and denominator of the original filter [cf.(2.2)] are real and positive quantities. These ill effects are due to the way of computing the power density spectra of signal and noise, which involves a subtraction of estimated spectra with inherent variabilities (see Figure 4.2). Although it has been shown (de Weerd and Martens, 1978) that this situation can be substantially improved by *smoothing* the estimated power density spectra it is obvious that at best only an approximation to the theoretical Wiener filter can be obtained.

The approximation becomes closer as the estimated spectra have smaller variability, that is, when the signal-to-noise ratio of the average becomes higher. In fact we thus arrive at a somewhat paradoxical element in the a.p.w.f. technique. An *a posteriori* filter is needed if the signal-to-noise ratio after averaging is still insufficient. But the lower the signal-to-noise ratio, the larger the variability of the estimated spectra becomes and hence the more unreliable the computed transfer function is. The only way to improve the reliability is by increasing the signal-to-noise ratio (e.g. by prolonged averaging), but this in turn decreases the need for using an *a posteriori* filter at all (see also Strackee and Cerri, 1977).

There have been other controversial points regarding the merits of a.p.w.f. as well. In particular the fundamental question has been raised whether it is in the absence of *a priori* knowledge on signal and noise by any means possible to improve beyond averaging (Strackee and Cerri, 1977; Nagelkerke and Strackee, 1979). It has recently been established, that the answer is affirmative, but a full discussion is rather beyond the scope of this paper. For further details the reader is referred to de Weerd (1981).

In view of the above we conclude that *a posteriori* "Wiener" filtering has been *inspired* by Wiener's theory, but that it deviates from this theory in major aspects [1]. The actual methodology is based on heuristic reasoning, rather than on mathematical rigor. Its application is restricted to deterministic signals with more or less time-invariant power distributions, contaminated with uncorrelated stationary noise, within a limited range of signal-to-noise ratios.

3. Which Filter to Use?

Two different filter transfer functions appear in the literature, namely, the filter originally formulated by Walter(1969):

$$\hat{H}_W(f) = \frac{\Phi_{ss}(f)}{\Phi_{ss}(f) + \Phi_{nn}(f)} \tag{3.1}$$

and a modified filter proposed by Doyle (1975):

$$\hat{H}_D(f) = \frac{\Phi_{ss}(f)}{\Phi_{ss}(f) + (1/N)\Phi_{nn}(f)} \tag{3.2}$$

where $\hat{H}(f)$ is the estimated transfer function (according to Walter and Doyle, respectively), $\Phi_{ss}(f)$ and $\Phi_{nn}(f)$ are the estimated power density spectra of the signal and the noise, and N the number of averaged ensemble elements. It has been shown (Doyle, 1975) that Walter's filter is the correct filter to use when "optimal"estimation of the signal in *single* ensemble elements is pursued, while Doyle's filter is the correct filter for use on the ensemble *average*.

It is, however, a misconception to state that "optimal" filtering of the ensemble average ultimately leads to the same result as averaging of the "optimally" filtered individual ensemble elements (Carlton and Katz, 1980). Ensemble averaging as well as a.p.w.f. are *linear* operators; hence their order of application to the input data can be interchanged (Figure 4.3). This obviously leads to the conclusion that if an "optimally" filtered ensemble *average* is desired, one can either average all ensemble elements and apply the filter once to the average (clearly the most economical way) or apply the *very same filter* to each ensemble element separately, after which these filtered elements are averaged. In either case, Doyle's filter is the correct filter to apply. The single correct application of Walter's filter lies in the "optimal" estimation of the

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[1]) For this reason the name of "Wiener" has deliberately been put between quotation marks.

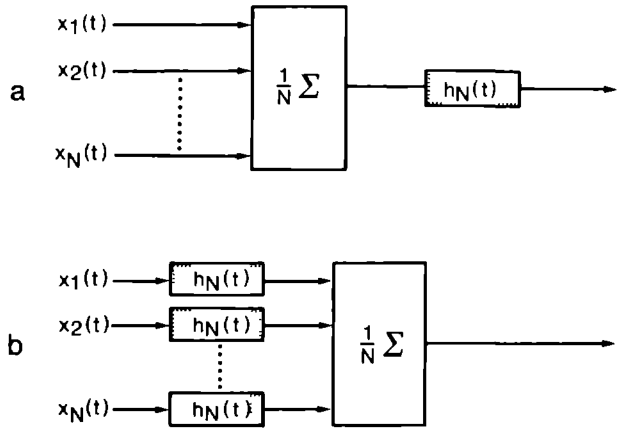

Figure 4.3a and b

The a posteriori *estimated "Wiener" filter (with impulse response $h_N(t)$) can be applied either to* a: *the ensemble average upon completion of the averaging procedure or to* b: *the individual ensemble elements prior to averaging. Since averaging and "Wiener" filtering are both linear operators, their order of application can be interchanged. Of course in either case the identical filter should be used*

signal within *single* ensemble elements *per se*. Subsequent averaging of these "optimally" filtered elements would lead to a nonoptimally filtered average.

To see why this is so, it suffices to investigate the limiting case where the number of averaged ensemble elements N tends to infinity. Walter's filter is a *fixed* filter, independent of N. When this filter is applied and the number of averaged ensemble elements is then increased, the variance of the estimate is further reduced, *but not the bias*. Therefore this estimator is *inconsistent*. That is, the estimate will not converge to the true value as N becomes large; there remains a systematic discrepancy. On the other hand Doyle's filter depends on N and when N increases, the transfer function of the filter tends to unity as should be expected when the average becomes virtually noise-free with large N. Hence the bias also vanishes; this estimator is consistent.

A slightly different and perhaps more instructive way of looking at the problem has been sketched in Figure 4.4. Here the two components making up the MSE, namely squared bias and variance [cf.(2.4)], have been drawn separately. On the abscissa an arbitrary measure for the degree of filtering has been assumed. Let us suppose now that we initially have an ensemble average of N ele-

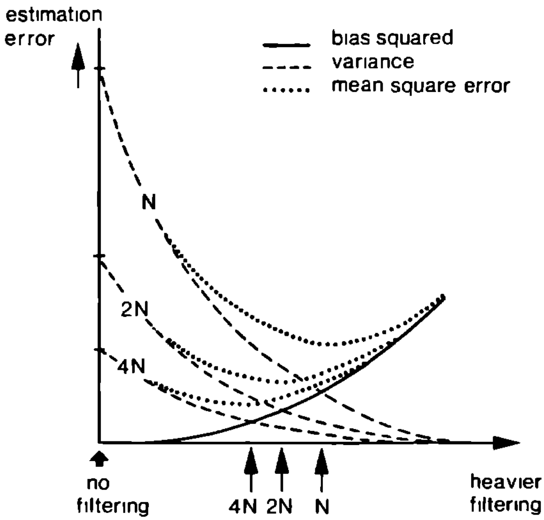

Figure 4.4
Optimal filtering of an averaged waveform implies minimization of the MSE, which is formed by the sum of the squared bias (solid line) *and the variance* (broken lines). *These components have been separately drawn as a function of an arbitrary degree of filtering (abscissa), and with the number of averaged ensemble elements (N, 2N, 4N) as parameter. Heavier filtering leads to an increased bias, less filtering leads to an increased variance. The compromise between the two is reflected by a minimum MSE. With prolonged averaging (from N to 2N or to 4N) the variance of the average decreases and the minimum of the MSE curve (vertical arrows N, 2N, and 4N) shifts to the left, indicating that the "optimal"* a posteriori *filters should be adapted accordingly*

ments to which an *a posteriori* filter is applied. With heavier filtering the variance of the average decreases, but at the same time the bias increases. The optimum degree of filtering (i.e. the theoretical Wiener filter) is found where the MSE reaches a minimum. If twice as many ensemble elements are then averaged, the variance of the average is halved, but with a fixed filter the bias remains the same. Clearly, the new minimum of the MSE curve is shifted. To obtain optimal filtering of the new average the filter should be adapted.

Of course, Figure 4.4 represents a largely schematized and idealized situation. In particular, as has already been explained in the previous section the minimum MSE is not attained in practice due to the fact that the transfer function is itself an estimate. Nevertheless, this figure can be helpful in illustrating some other practical facts. For instance, it can readily be seen from the diminishing distance between the *minimum* MSE and the MSE obtained without filtering, that as the number of averaged ensemble elements N increases, a.p.w.f. becomes continually less effective. Also it can be verified that for small N this distance can be large and thus the improvement after application of a.p.w.f. considerable due to the large reduction in variance. But at the same time the bias may have become unacceptably high, a point which we will briefly deal with in section 6.

4. Comments on Recursively Computed Estimates

It has been suggested (Carlton and Katz, 1980) that recursive schemes for "optimal" filtering of evoked potentials might prove more effective, although the algorithms actually proposed failed to yield reliable results. We want to point out, however, that such procedures are in fact inadmissible.

Basically there are two main objections. By using as input to the filter the previously filtered sum plus the most recently measured response as suggested by Carlton and Katz (1980), previous responses are, in effect, *cumulatively* filtered. This implies that subsequently estimated signals will gradually become smaller since the magnitude of the filter transfer functions is always smaller than (or equal to) unity. Moreover, as a consequence of the actual procedure as described in detail by Carlton and Katz (1979), the transfer functions themselves are also systematically underestimated, as will be explained here. From (3.2) and Figure 4.2 it follows that Doyle's filter, expressed in terms of the estimated spectra $\Phi_{\overline{xx}}$ and $\overline{\Phi_{xx}}$ reads

$$\hat{H}(f) = \frac{N}{N-1}\left(1 - \frac{1}{N}\,\frac{\overline{\Phi_{xx}}(f)}{\Phi_{\overline{xx}}(f)}\right) \qquad (4.1)$$

The denominator of the expression between brackets contains $\Phi_{\overline{xx}}(f)$, the power density spectrum of the ensemble average. The quintessence of Carlton and Katz's recursive computation scheme is to replace that spectrum by the power density spectrum of the previously filtered sum [1)] plus the most recently

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1) Carlton and Katz's formulation of the filter transfer function departs from *summated* instead of *averaged* spectra, but this makes, of course, no essential difference.

measured response. The latter spectrum, properly corrected for the total number of responses, gradually becomes smaller in magnitude than the spectrum of the ensemble average, for the same reasons as stated above. From (4.1) it is then easily verified, that the filter transfer function gradually becomes more underestimated.

The two effects mentioned above obviously amplify each other and explain the finding that the recursive filtering scheme actually used, ultimately depresses the estimates to zero (Carlton and Katz, 1980).

5. Usefulness in Average Evoked Potential Estimation

In the preceding sections it has been shown that in fact only Doyle's filter (3.2) and (4.1) is worth further consideration. In this section we address the question of whether the application of this filter is useful in estimating average evoked potentials.

Why did Carlton and Katz (1980) find that application of Doyle's filter did not consequently improve their estimates, although other authors obtained "highly improved" estimates (Hartwell and Erwin, 1976) or found a.p.w.f. to be "extremely valuable" (Kearney, 1979)? Disregarding some aspects which are for the moment of lesser importance, such as the modifications that have been suggested to increase the effectiveness of a.p.w.f. (de Weerd and Martens, 1978), the major reason that Carlton and Katz found no improvement lies beyond a doubt in the fact that highly *transient* signals were chosen for the evaluation of the method.

In expanding upon this statement we will follow a more intuitive reasoning. First of all, it should be noted that the spectral power density of a transient signal depends on the observation interval, a point that was already discussed by Ungan and Basar (1976). These authors essentially pointed out that if a signal is time limited, then its spectral power density decreases with increasing observation interval (Figure 4.5). More precisely, if a signal $s(t)$ is time limited about the interval $(0,\tau)$ then its Fourier transform $S(f)$ is given by

$$S(f) = \int_0^\tau s(t)e^{-2\pi ift}dt \qquad (5.1)$$

which is independent of the observation interval T (provided that $T \geq \tau$). However, the power density spectrum defined by

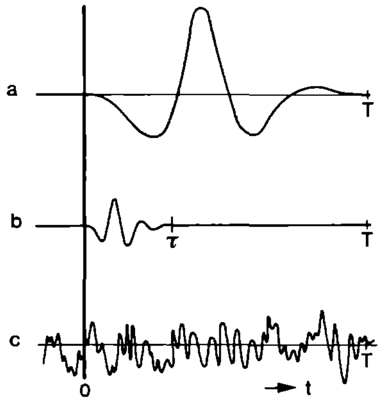

Figure 4.5a–c
It can be argued that the applicability of a posteriori *"Wiener" filtering is limited to signals with a time-invariant power distribution. But averaged evoked potentials usually are of a more complex character and, conceptually, consist of different components. Two such components are shown schematically in the example above.* a: *Low-frequency part of relatively long duration.* b: *Higher frequency part of shorter duration. Particularly in the latter case it is obvious that for a correct description of the signal power not only its* spectral, *but also its* temporal *distribution should be taken into account.* c: *No such problems appear with respect to the noise, as long as this can be assumed to be stationary. T represents the observation interval of the evoked potential*

$$\Phi_{ss}(f) = \frac{1}{T} \left| \int_0^T s(t)e^{-2\pi ift}dt \right|^2 = \frac{1}{T} |S(f)|^2 \tag{5.2}$$

clearly decreases inversely proportionally with the observation interval T. On the other hand, the expected value of the power density spectrum of the noise $\Phi_{nn}(f)$ does not depend on T, provided that the noise is stationary.

Now in practice the observation interval T is usually chosen so as to cover the entire duration of the evoked potential waveform, or at least the relevant part of it. This duration is mainly determined by the low-frequency part ("slow waves") of this signal (Figure 4.5a). However, the early part of an evoked potential, just beyond the stimulus, usually contains transient components of much higher frequencies and shorter durations (Figure 4.5b). The power density spectrum provides an average power description over the full observation interval. Essentially, it is interpreted as the power distribution of continually present wavetrains. If a wavetrain is present only a *part* of the observation interval, like in Figure 4.5b, its power contribution is smeared out over the *entire* interval. As a consequence, the power is underestimated in the time interval that the wavetrain is present, while it is of course overestimated in the interval that the wavetrain is absent. Clearly, when dealing with transient signals an adequate power description (and consequently a "Wiener" filter description based on it), requires that the *temporal* distribution of this power is taken into consideration as well. We will come back to this point later.

The above reasoning explains why application of a.p.w.f. to highly transient signals corrupted with noise has shown to be ineffective (Carlton and Katz, 1980). In this respect the actual choice of the *somatosensory* evoked potential for evaluation of the "Wiener" filter is quite unfortunate, a point already made by de Weerd and Martens (1978). Auditory brainstem evoked potentials and, to a lesser extent, evoked potentials from the visual modality may actually show a somewhat more periodic character (Figure 4.6). Application of a.p.w.f. in these fields may occasionally prove more effective, depending on how time-invariant the power distribution of the signal actually is.

There exists, however, a more powerful method of estimation, namely *time-varying filtering,* which has been specially developed for application to transient evoked potentials (de Weerd et al, 1977; de Weerd, 1981; de Weerd and Kap, 1981). Essentially, the method can be seen as a generalization of the *a posteriori* "Wiener" filtering, in which the transfer function $\hat{H}(f)$ has been superseded by a weighing function in the combined time-frequency domain. Thus the temporal distribution of the signal-to-noise power ratio is taken into account as well; a matter of major importance when dealing with transient signals, as pointed out previously. This technique has proven its superiority to a.p.w.f. for a vast number of auditory, somatosensory, and visual evoked potentials, and its application in these fields is therefore highly recommended [1].

6. Discussion

Several papers dealing with the technique of a.p.w.f. have created the impression that this technique provides an "optimal" estimate of the "true" evoked potential, no matter what characteristics the true signal may have. On the

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[1] A simple to use software package for time-varying filtering that can be interfaced with virtually any current software averaging program is available from the author.

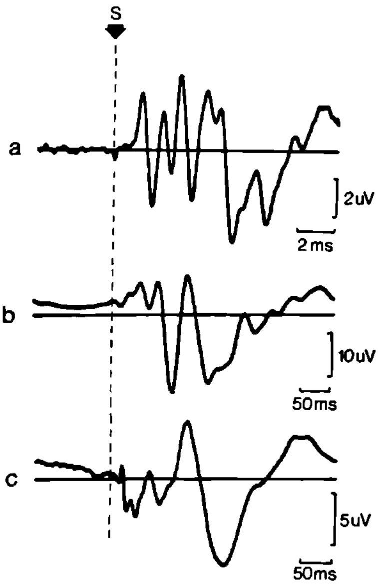

Figure 4.6a–c
Evoked potentials of different modalities may show distinct characteristics, also depending on the time base on which they are observed. For instance, on the usual time bases auditory brainstem evoked potentials (a) *show a somewhat periodic waveshape, which is observed less in visual evoked potentials* (b). *Somatosensory evoked potentials* (c), *on the other hand, usually show pronounced transient components. S: stimulation moment. Note also differences in calibration bars*

other hand, there have been papers, particularly recent ones, which have criticized various aspects of the method.

It appears that there are indeed several problems, mainly due to the fact that 1) the filter transfer function is itself an estimate, with an inherent bias and variance that become large when a filter is most needed, i.e. at low signal-to-noise ratios; 2) a.p.w.f. is limited to deterministic signals with an (almost) time-invariant power distribution; and 3) in spite of its seeming simplicity the method has several problematic aspects some of which we have tried to clarify above.

In spite of these problems, which clearly limit the general applicability of the method, it has been shown for various signals that a.p.w.f. may nevertheless lead to a better estimation (in the MSE sense) than averaging provides (Hartwell and Erwin, 1976; de Weerd and Martens, 1978; Kearney, 1979). Quantitative measures of this improvement are difficult to produce, since these depend on the signal-to-noise ratio of the ensemble average and the amount of overlap between the power density spectra of the signal and the noise. However, simulation studies with relatively nontransient signals (de Weerd and Martens, 1978; Kearney, 1979) have shown that the MSE could be reduced after application of a.p.w.f. by a factor approximately ranging from one to five. With very low signal-to-noise ratios this factor may even become substantially larger, but it has been pointed out (de Weerd and Martens, 1978) that in spite of seemingly dramatic "improvements" the estimate may readily become an unacceptably distorted image of the true underlying signal which is also affected by the filter.

In the light of the above conclusions the most crucial question which remains to be discussed is whether evoked potentials can reasonably be assumed to be nontransient. Many times, such an assumption appears hardly justifiable, although some periodic features - particularly in auditory brainstem and to a lesser extent in visual evoked potentials - cannot be denied. Hence, in the latter cases a.p.w.f. may sometimes be useful. However, the more powerful technique of time-varying filtering has proven to be superior to "Wiener" filtering in estimating evoked potentials (de Weerd, 1981). It therefore appears more advisable to apply time-varying filtering when dealing with these problems.

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Chapter 5

Spectro-temporal Representations and Time-varying Spectra of Evoked Potentials

A Methodological Investigation

J.P.C.M. de Weerd and J.I. Kap

Biol. Cybernetics, in press (1981)

Abstract

Evoked potential waveforms are generally of a dynamic, transient character. Consequently, their spectral energy distribution cannot be adequately described by time-*in*variant representations, such as the power density spectrum. Obviously, a *spectro-temporal* description is needed. Appropriate means for obtaining such descriptions are discussed, on the basis of theoretical considerations concerning simultaneous time-frequency representations and methods of *short-time spectral analysis*. With reference to the "uncertainty principle", particular attention is paid to time-bandwidth products of various filter types, used in relation with the latter technique. It is concluded that the method of bandpass filtering with proportional bandwidth filters, having cosine transfer functions, arises as a suitable solution in evoked potential analysis. The results of applying this method to somatosensory, visual, and brainstem auditory evoked potentials are presented.

1. Introduction

From the physical point of view, evoked potentials elicited by brief stimuli can be considered as impulse responses from electrophysiological systems. Typically, their waveforms are of a complicated, transient character. One might think of these waveforms as being compositions of damped oscillations of different frequencies (cf. Basar and Ungan, 1973). This idea is illustrated in Figure 5.1, which shows a somatosensory evoked potential (average of 200 sweeps), passed through a bank of overlapping, zero-phase, bandpass filters. Clearly, the outputs of successive filters show waveforms which become shorter in duration as the frequency increases. This is especially clear in the bottom trace, where it appears that the waveform of relatively high frequency and short duration is largely responsible for the "early components" in the evoked potential as a whole. Later we will deal with these aspects in a more quanti-

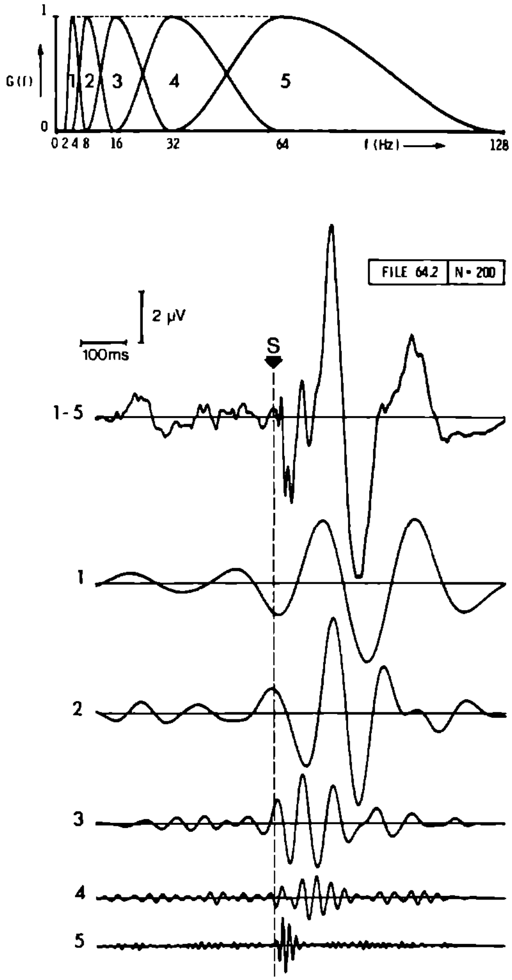

Figure 5.1
Illustrative example of a somatosensory evoked potential (average of 200 sweeps; trace denoted by '1-5') *and its various constituting components (traces denoted* 1,2,3,4 *and* 5) *obtained by passing the evoked potential through a bank of bandpass filters shown in the upper part of figure. Note the approximately inversely proportional relationship between the mean frequency and the duration of the waveforms in the various frequency bands*

tative manner.

Although evoked potentials are ordinarily obtained as time functions, it may, for various reasons, be desirable to describe their waveform, or more specifically their energy distribution, in the frequency domain. It has, for instance, been suggested (Walter, 1969; Nogawa et al., 1973; Doyle, 1975; Hartwell and Erwin, 1976), to use estimated power density spectra of the evoked potential and of the background activity for the calculation of a "Wiener" filter, in order to improve the waveform beyond averaging. However, in view of the transient time domain structure of evoked potentials it cannot be expected that time-*in*variant descriptors such as the power density spectrum provide an adequate frequency representation. The reason for this is that the power density is an *average energy* measure [1], as a function of frequency, over the *entire* observation interval. If a signal is present during only a part of that interval (see e.g. the bottom trace of Figure 5.1), then its energy is thought to be smeared out over the entire interval, which may lead to meaningless results (see Ungan and Basar, 1976 and van der Tweel et al., 1980 for a similar discussion). This is one of the major reasons why the application of "*a posteriori* 'Wiener' filtering" to averaged evoked potentials has been rightly criticized.

Obviously, the spectral energy distribution of an evoked potential waveform calls for a time-varying representation. In other words, a *spectro-temporal* description of the energy distribution is required. The purpose of this paper is to establish appropriate means for obtaining such a representation, whereby the relative emphasis is on practical feasibility, rather than mathematical elegance. At the same time this study forms the basis for the newly introduced *time-varying filtering* technique (de Weerd et al., 1977; de Weerd, 1981; de Weerd and Kap, 1981). We start with a theoretical analysis of the energy distribution of a signal in the combined time-frequency plane (section 2). This leads to the conclusion that the method of short-time spectral analysis which uses a bank of proportional bandwidth filters is most appropriate in the evoked potential field. In section 3 the desired properties of such bandpass filters are investigated and suitable characteristics for their transfer functions established. This section is mainly based on the theory of analytic signals and the uncertainty principle. Finally in section 4 some typical results of spectro-temporal representations are presented for various evoked potential waveforms.

2. Description of Signal Energy in the Time-frequency Plane

A time-varying power density spectrum is essentially an energy description in time and frequency *simultaneously*. Such a description may be obtained in a variety of ways. From the mathematical point of view the most elegant method is certainly a description in terms of the "complex energy (double) density" function (Rihaczek, 1968). Although such a description leads to considerable theoretical insight, it will be shown that it is not a very practical one (section 2.1).

Instead, a more classical approach may be followed, namely "short-time spectral analysis", as employed for instance in real-time spectral analysers.

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[1]) Note that the physical dimension of a power density spectrum (V^2/Hz) is that of an energy (V^2s)

Digitally operated systems, based on this principle, either make use of a bank of bandpass filters or a bank of time windows (section 2.2). In practice these systems should be matched to the time-frequency structure of evoked potential waveforms. We will discuss in which way this can be realized, on the basis of *a priori* knowledge of the general characteristics of these waveforms (section 2.3).

2.1 The Complex Energy (double) Density Function

The spectral energy distribution of a signal x(t) which is either deterministic and periodic, or random and stationary, or a combination of both, can be properly described by the power density spectrum. This spectrum is the Fourier transform of the signal's autocorrelation function, which depends only on the time *difference* between two successive observation moments. When dealing with a signal which is *transient* or *nonstationary*, it also becomes important at which particular moment the signal is observed.
We therefore define the time-dependent autocorrelation function:

$$\Psi_{xx}(t,\tau) = E\{x(t)\ x(t+\tau)\} \qquad (2.1)$$

where t represents the actual observation time, τ the time difference between two observation moments and the symbol E{ } means expectation taken over the ensemble. By taking the Fourier transform of $\Psi_{xx}(t,\tau)$ with respect to τ we obtain the *time-dependent power density spectrum*

$$P_{xx}(t,f) = E\{x(t)\ X(f)\ \exp(2\pi ift)\} \triangleq E\{e(t,f)\} \qquad (2.2)$$

where X(f) is the Fourier transform of x(t). Note that, in general, both $P_{xx}(t,f)$ and e(t,f) are *complex* functions of time and frequency.

It should be pointed out at once that a multitude of definitions and different terminology with respect to the time-dependent power density spectrum exists. For example, the real part of the function e(t,f) has been referred to as the "time-frequency energy density distribution" (Ackroyd, 1970), while the real part of $P_{xx}(t,f)$ has been termed the "mean instantaneous power spectrum" (Levin, 1964). However, other definitions have also been proposed (e.g. Fano, 1950; Schroeder and Atal, 1962; Ackroyd, 1973).

It appears that a unified and mathematically elegant approach is possible if we depart from the *analytic signal* instead of the *real* signal (see appendix). Following Rihaczek (1968), who defines the time-dependent autocorrela-

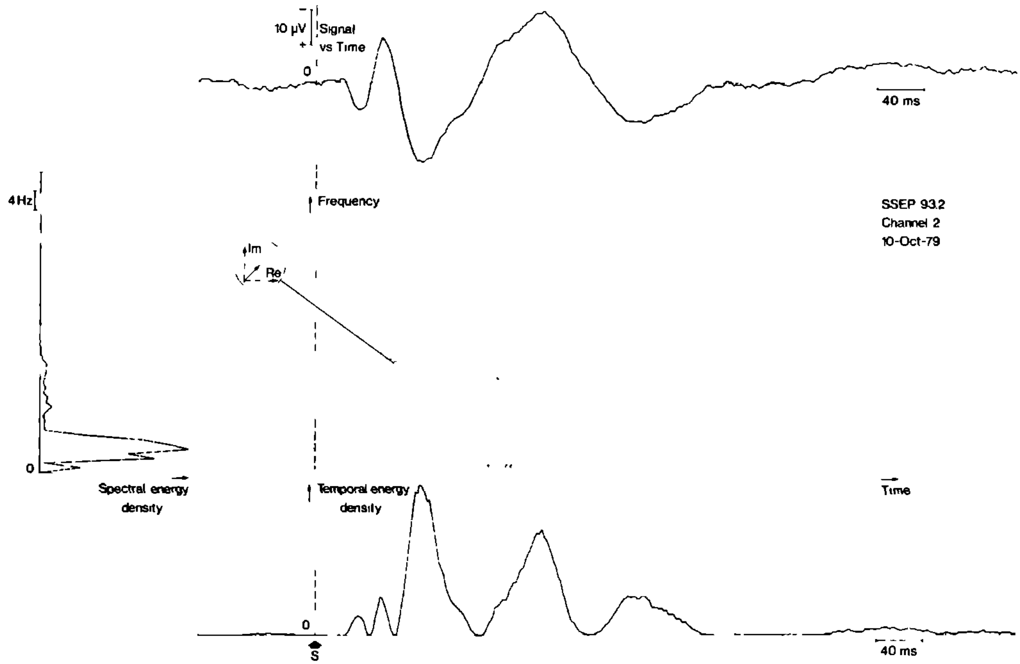

Figure 5.2

Representation of signal energy in the time, frequency and combined time-frequency domain. Upper part: *a somatosensory evoked potential (average of 128 sweeps); vertical line marked by S indicates the moment of stimulation.* Center: *simultaneous spectro-temporal description of the signal's energy, using the Complex Energy (double) Density function* (CED). *The CED is a* complex *function, whose value in each point of the time-frequency plane has been represented by a vector, showing the magnitude and the phase at that point.* Lower part: *integration of the CED over frequency gives the temporal energy density, or instantaneous power vs. time.* Middle left: *integration of the CED over time gives the spectral energy density (usually called the energy density spectrum). Integration over both frequency* and *time gives the total signal energy*

tion function in a slightly different manner, the function e(t,f) from (2.2) then converts into:

$$\varepsilon(t,f) = \xi(t)\ \Xi^*(f)\ \exp(-2\pi ift) \tag{2.3}$$

where $\xi(t)$ represents the analytic signal of $x(t)$, $\Xi(f)$ the Fourier transform of $\xi(t)$, and the starring denotes the complex conjugate. The function $\varepsilon(t,f)$ has been referred to as the "complex energy density" function (Rihaczek, 1968). Inspection of (2.3) shows, that the physical dimension of $\varepsilon(t,f)$ is that of an *energy* (V^2s). Therefore, it would perhaps be more appropriate to call $\varepsilon(t,f)$ a *double density* function (namely a complex energy distribution per unit of time and per unit of frequency) or, concisely, a complex energy distribution function *(CED)*.

Since $\varepsilon(t,f)$ is a *complex* function of time and frequency, a particular value of $\varepsilon(t_o,f_o)$ in the time-frequency plane is a complex number which can be represented by a vector with a modulus (the magnitude at that point) and an argument (the phase), (cf. Johannesma and Aertsen, 1979). An example of a CED representation of a somatosensory evoked potential is shown in Figure 5.2, together with the conventional energy density spectrum and the instantaneous temporal power. The remarkable aspects of the CED are, that this function apparently preserves phase information, while it also appears not to suffer from the inherent loss of time and/or frequency resolution which is encountered in other forms of short-time spectral analysis (see section 2.2).

However, the fact that the CED is a *complex* function, indicates already that it cannot represent a *physically meaningful*, i.e. real and positive energy distribution in time and frequency. Neither can the real part of the functions $\varepsilon(t,f)$ nor $e(t,f)$, since negative energies occur as well. It should be stressed that in principal the same remarks apply to the time-dependent power density spectrum $P_{xx}(t,f)$ defined by (2.2). The interpretation of negative or, more generally, complex energy has been discussed by Page (1952), Mark (1970), and Ackroyd (1970, 1973). In these discussions a direct connection is made to the *uncertainty principle* (Gabor, 1946; see section 3.1), which states that a signal cannot be localized within an arbitrarily small region in time and simultaneously within an arbitrarily narrow band in frequency. Several authors (e.g. Rihaczek, 1968; Mark, 1970; Ackroyd, 1970) have pointed out that a physically meaningful energy distribution can only be obtained by integrating the CED over a suitable area in the time-frequency plane or, equivalently, by smoothing this function along the time and frequency axes [1)].

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1) Mark (1970) therefore termed such a smoothed energy distribution function "the physical spectrum".

Mark (1970) and Ackroyd (1971, 1973) showed that such a smoothing operation corresponds with a double (i.e. two-dimensional) convolution of the CED of the actual signal and the CED of the smoothing window. The effective time duration and bandwidth of that window determine the temporal and spectral resolution of the resulting time-frequency energy distribution. Ackroyd (1970, 1971, 1973) further clarified the relation between the smoothed CED and more classical approaches of short-time spectral analysis, (see section 2.2), as well as other time-frequency energy distribution functions, such as the "instantaneous power spectrum" (Page, 1952; Turner, 1954) and the "short-time power spectrum" (Fano, 1950; Schroeder and Atal, 1962). All these methods can be seen either as *approximations* or, alternatively, as physically meaningful estimates of some underlying "true" time-frequency energy distribution.

The disadvantage of the approximating methods is that they all make use of some *a priori* chosen time or frequency window. Thus the spectral and temporal resolution are also determined *a priori* and, unlike computation via the CED, cannot be adapted to the data *a posteriori*. Nevertheless, in practice some of these methods are clearly preferable for reasons of computation-time and storage requirements. The following argument easily demonstrates this fact. If a signal has a duration of T seconds and a bandwidth of W Hertz, a minimal sampling scheme will produce 2WT (real) data points in the time or, equivalently, in the frequency domain. The CED matrix, computed for *positive frequencies* only, will thus contain $2(WT)^2$ elements (cf. (2.3) and Figure 5.2), and requires the same order of magnitude multiplications. The smoothing operation requires an even larger number of multiplications in addition to this. On the other hand, short-time spectral analysis as discussed in the next subsection requires 2WT data points only and a similar order of magnitude multiplications. Therefore it is an unnecessary expense to follow the CED approach, if we have sufficient *a priori* information concerning the signal structure in order to be able to choose the required spectral and temporal resolution beforehand. This problem will be discussed in section 2.3. Prior to this we will introduce two more commonly used methods for short-time spectral analysis.

2.2 Methods of Short-time Spectral Analysis

In the present context two methods of short-time spectral analysis appear to be relevant. These methods either use a bank of time windows, or a bank of bandpass filters. Of course several other techniques exist (such as the "heterodyne" method), but these do not lend themselves easily to implementation on a digital computer and will not be dealt with further.

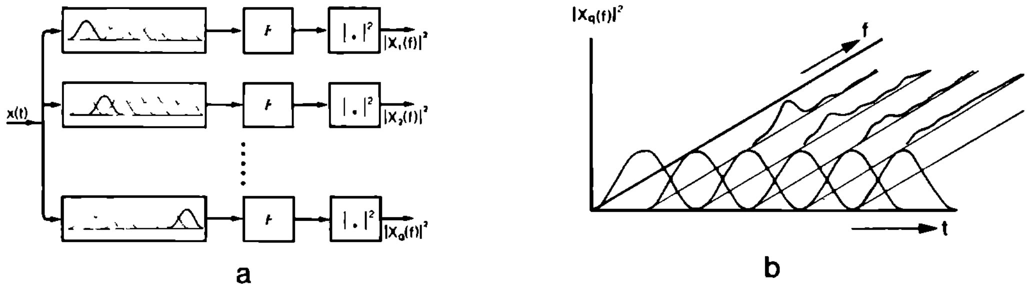

Figure 5.3a and b
Short-time spectral analysis by time sectioning.
a: *The input signal is sectioned into time segments which are Fourier transformed. The squared magnitudes of these transforms provide sequential energy density spectra.* b: *These spectra can be regarded as cross sections in time of the spectro-temporal energy distribution*

2.2.1 Short-time Spectral Analysis by Time Sectioning

In this method the total observation time of the waveform is segmented into (possibly overlapping) smaller time epochs, from which energy density spectra are computed in the usual way (Figure 5.3). In the field of digital signal processing this is the more commonly employed method of time-varying spectral analysis, since we indeed obtain sequential energy density spectra as time proceeds. The time segmentation is carried out through multiplication of the original waveform by a moving time window, the width of which should be matched to the structure of the waveform under consideration. Clearly, when rapid changes in the waveform are to be resolved a high temporal resolution, i.e. a narrow time window is necessary. But this is at the cost of spectral resolution, which is inversely related to the time width (cf. section 3).

2.2.2 Short-time Spectral Analysis by Bandpass Filtering

An alternative method of short-time spectral analysis is by using a bank of (possibly overlapping) bandpass filters with sequentially increasing center frequencies. When the (time domain) output of the individual filters is squared, we obtain a slowly varying component, i.e. the desired time-varying (instantaneous) power with, superimposed on it, a rapidly varying component with a frequency around twice that of the center frequency of the bandpass filter under consideration. The latter component is usually removed by appropriate smoothing or lowpass filtering. This approach is often used in dynamic or real-time spectral analyzers.

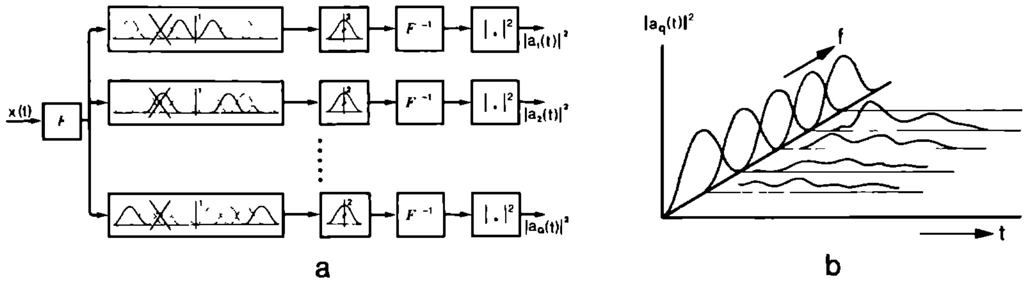

Figure 5.4a and b

Short-time spectral analysis by bandpass filtering. a: *The input signal is Fourier transformed and sectioned in frequency. Negative frequencies are omitted and positive frequencies within the passbands doubled in magnitude and translated to zero frequency. Inverse Fourier transformation gives the complex envelopes, whose squared magnitudes provide the instantaneous power, or temporal energy density, in subsequent passbands.* b: *These squared envelopes can be regarded as cross sections in frequency of the spectro-temporal energy distribution*

However, as shown in the appendix, the mathematical concepts of the "analytic signal" and the "complex envelope" can be used to advantage in this type of spectral analysis (Figure 5.4). The procedure is as follows. By omitting the negative frequencies and doubling the magnitude of the positive ones in the Fourier transform of each bandpass filter output, we obtain the spectrum of the *analytic signal*. When this spectrum is translated towards zero by an amount equal to the center frequency of the filter, the spectrum of the *complex envelope* is obtained. The squared magnitude of the complex envelope itself is a direct measure for the instantaneous power of the signal in the corresponding passband (cf. Aertsen and Johannesma, 1980). This implies that in principle no additional smoothing or lowpass filtering is necessary, although this should be done when dealing with random signals in order to reduce the variance of the power estimates (see e.g. Jenkins and Watts, 1968).

The above procedure is equivalent to the method of "complex demodulation", introduced into EEG analysis by Walter (1969). In spite of Walter's seemingly different approach of using "local oscillators", the above method measures essentially the same quantity, namely the instantaneous power (or amplitude) in a particular frequency band. Of course this form of short-time spectral analysis is subject to the same limitations with respect to temporal and spectral resolution as is the time sectioning method. In fact, any short-time spectral energy measurement requires a minimum effective bandwidth and observation time, and frequency shifts within this bandwidth, as well as power variations within this observation time, remain obscured. This fundamental principle, which is further elaborated in section 3, will play an important role in our final choice of a practical method for short-time spectral analysis.

2.3 The Time-frequency Structure of Evoked Potentials and Short-time Spectral Analysis

It has been shown (Ackroyd, 1971) that the two methods of short-time spectral analysis, discussed in the foregoing subsection, can be interpreted as a smoothing of the CED function $\varepsilon(t,f)$ along the time and frequency axes. However, at present no systematic methods seem available to indicate how these procedures should be further elaborated in practice, that is, how the bank of time or frequency windows should be chosen for the actual problem at hand. Ackroyd (1970, 1973) even stated that these questions cannot be answered other than by intuition, experience and subjective judgement. With these thoughts in mind we will try to summarize some characteristic features of the general evoked potential structure in time and frequency.

Inspection of various evoked potential waveforms from different modalities (Figure 5.5) shows that the "mean frequency" of the signal has a general tendency to decrease with increasing lag-time beyond the stimulus. (This fact was also considered by Coppola et al. (1978) in simulating a data base of single trial evoked potentials.) Stated in another way, it appears that evoked potential waveforms consist of several components which become broader, i.e. of lower frequency, with increasing lag-time. This is particularly clear in somatosensory evoked potentials (Figure 5.5a; see also Figure 5.1). This type of evoked potential consists of "early" components of relatively high frequency and short duration, and later components of lower frequency and longer duration. A similar structure can be observed in visual evoked potentials (Figure 5.5b). In auditory brainstem evoked potentials, the transition from high frequency to lower frequency components is somewhat more gradual (Figure 5.5c) [1].

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[1]) Note that if we were to observe the whole auditory evoked potential, including the middle and late components, this e.p. would probably appear to be the most pertinent example of our statement. Indeed the normal auditory evoked potential shows some resemblance to a sinusoidal waveform if plotted on a *logarithmic* time base (See Picton et al., 1974; their Figure 1B).

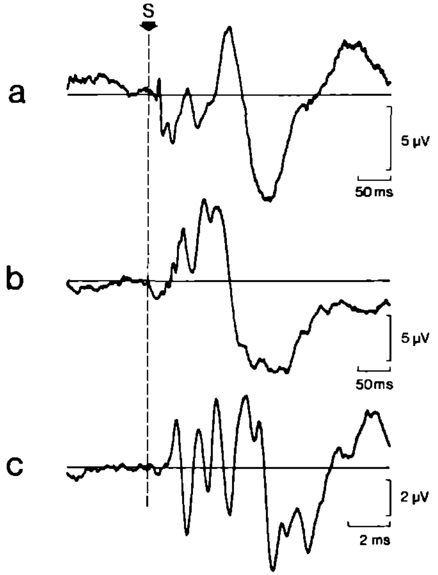

Figure 5.5a-c
Typical examples of evoked potentials from the human brain. a: *Somatosensory evoked potential (C3-A1; average of 256 sweeps; same subject as in Figure 5.1) after supramaximal stimulation of the right median nerve at the finger;* b: *Visual evoked potential (Cz-A2; average of 256 sweeps) after binocular stimulation with bright light flashes;* c: *Brainstem auditory evoked potential (Cz-M1; average of 2048 sweeps) after binaural click stimulation (70 dB SL).* S: *moment of stimulation. Note differences in calibration bars*

These observations suggest that an adequate short-time spectral description of evoked potentials requires time or frequency windows of *variable*, rather than *fixed*, width. For instance, if we would choose the time sectioning method (section 2.2.1), this would imply the use of windows which successively increase in duration with increasing lag-time beyond the stimulus. However, this procedure would require *a priori* knowledge about the *time structure* of the evoked potential waveform, specifically with respect to the latency and duration of various components. But in an evoked potential investigation it is exactly the latencies of various components which are clinically the most important, but of course *unknown*, parameters, from which it is well known that these may vary widely in pathological situations. The time sectioning method seems therefore not well suited to our problem.

On the other hand the bandpass filtering method seems more applicable. In this case we should, analogously to the time sectioning method, use a bank of bandpass filters with increasing bandwidth as the center frequency increases. This ensures that components of short duration, and high frequency, are captured with a large bandwidth (so that they can be accurately located in time), while components with long duration, and low frequency, are captured with a small bandwidth (so that they can be accurately located in frequency). The use of nonuniform filters is further motivated by the *frequency structure* of evoked potential waveforms, which generally shows spectral peaks with a roughly constant *relative* bandwidth (see e.g. Basar et al, 1975). Conceptually, this could be caused by the activity of several damped oscillatory systems, each having a specific resonance frequency and a bandwidth proportional to this frequency. A property of such systems is that the duration and the mean period of the system's impulse response are proportionally related. Sciarretta and Erculiani (1975) noted this phenomenon in analyzing *spontaneous* EEG, which brought them to propose an alternative Fourier transform, which they called the "BERG transform". The essential feature of it is that the duration of the time epoch over which the transform is carried out depends inversely on the frequency of the spectral component which is analyzed. This approach can be seen as a step towards a short-time spectral analysis scheme that is well matched to the data under consideration.

The approximately constant relative bandwidth property suggests the use of a filterbank, which also possesses this property, namely a bank of filters having a bandwidth proportional to their center frequency. A first practical choice then is a bank of *octave* filters. Of course, other banks can be chosen as well, depending on the particular problem at hand. However, in applying a bank of octave filters, as part of the "time-varying filtering" method, it was found that this simple bank provides a good compromise between the desired resolution in time and frequency (cf. de Weerd and Kap, 1981 for further discussion). In this respect the shape of the *transfer function* of individual filters in the bank is also of considerable importance. This is a consequence of the fact that good temporal *and* spectral resolution in short-time spectral analysis can only be obtained by using filters with a small time-bandwidth product. In the next section, we will investigate this problem in more detail.

3. Choice of Bandpass Filter Transfer Function

In the preceeding section we have already mentioned that short-time spectral analysis cannot provide both arbitrarily large temporal and spectral resolution at the same time. This is explained by the fact that no physical time or frequency window can have arbitrarily small width in time and frequency simultaneously. For a more rigorous approach to this principle we need operational definitions of (effective) time duration and bandwidth. Of the many different definitions which exist, we will choose the rather widely adopted ones introduced by Gabor (1946) in a, by now, classical paper. In this paper Gabor also derived a lower limit for the time-bandwidth product, which became widely known as the "uncertainty principle" (section 3.1). In applying Gabor's theory to *bandpass filters*, the analysis is greatly simplified by introducing the "equivalent lowpass filter", which is essentially a frequency translated version of the original bandpass filter, and is directly related to the concepts of the analytic signal and the complex envelope (section 3.2). On the basis of these concepts, the time-bandwidth product of various bandpass filter types is investigated (section 3.3). It is shown that the choice which might seem perhaps most obvious, namely a filter with rectangular transfer function, is in fact a totally unsatisfactory one. In section 3.4 we derive a "practical" time-bandwidth product, from which the temporal resolution at a particular filter output can be calculated, once the type and bandwidth of the filter are specified. Finally, since our short-time spectral analysis is performed on a digital computer, we are dealing with sampled signals. In the last subsection we therefore generalize previously obtained results for the discrete case.

3.1 The Uncertainty Principle

We start with a brief summary of the essential definitions of signal duration, bandwidth, and the uncertainty relation, based on the work of Gabor (1946). Following Gabor's original arguments, we thereby depart from the *analytic signal* (see the appendix), instead of from the (real) signal itself. It should be stressed that this is only a matter of convenience. The measures for signal duration and bandwidth which we finally arrive at, apply equally well to the original real signals. Denoting the analytic signal by $\xi(t)$ and its Fourier transform by $\Xi(f)$, the *signal energy* I is defined as [1]

$$I = \int|\xi(t)|^2\, dt = \int|\Xi(f)|^2\, df \tag{3.1}$$

where the second equality is due to Parseval's theorem.

The *mean "epoch"* of a signal is defined as

$$\mu_t = \frac{1}{I}\int t|\xi(t)|^2\, dt \tag{3.2}$$

and the *mean-square signal duration* as

$$\sigma_t^2 = \frac{1}{I}\int t^2|\xi(t)|^2\, dt - \mu_t^2 \tag{3.3}$$

Analogously, the *mean frequency* of a signal is defined as

$$\mu_f = \frac{1}{I}\int f|\Xi(f)|^2\, df \tag{3.4}$$

and the *mean-square bandwidth* as

$$\sigma_f^2 = \frac{1}{I}\int f^2|\Xi(f)|^2\, df - \mu_f^2 \tag{3.5}$$

For our purposes the quantities σ_t and σ_f, which we will refer to as the *rms*

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[1]) Integration limits are assumed $(-\infty,\infty)$ throughout.

duration and *rms bandwidth* are particularly important.
Gabor (1946) has shown that the product of these two must satisfy the following relation

$$\sigma_t \sigma_f \geq \frac{1}{4\pi} \tag{3.6}$$

known as the *uncertainty principle*. The equality only holds for signals having Gaussian envelopes in time and frequency. A more detailed discussion on this point is provided in the following subsections.

It should be noted that (3.3) and (3.5) represent mean-square, rather than *effective*, values for duration and bandwidth. Apparently there is some confusion on this point. For instance, Gabor (1946) defined the quantities $\sigma_t\sqrt{2\pi}$ and $\sigma_f\sqrt{2\pi}$ as the *effective* duration and bandwidth respectively, while Rihaczek (1969) suggested taking $2\pi\sigma_t$ and $2\pi\sigma_f$. These differences in definition become important when actual figures for bandwidth and duration must be specified in practice and we will come back to it in section 3.4.

3.2 Some Properties of Bandpass Filters

For application of the previously described equations we introduce the analytic signal representation for bandpass filters by defining the following *system functions* (see also appendix):

$h(t)$: (real) impulse response of bandpass filter
$H(f)$: filter transfer function; Fourier transform of $h(t)$
$\lambda(t)$: analytic impulse response
$\Lambda(f)$: Fourier transform of $\lambda(t)$; equals $2H(f)$ for $f>0$ and 0 for $f<0$
$h_c(t)$: complex envelope of $h(t)$
$H_c(f)$: Fourier transform of $h_c(t)$; frequency translated version of $\Lambda(f)$

The introduction of these functions will greatly simplify a further analysis as will soon become obvious. From the appendix, (A6), (A1) and (A9), it appears that the following additional relations exist:

$$h(t) = \mathrm{Re}\{\lambda(t)\} = \mathrm{Re}\{h_c(t)\exp(2\pi i f_o t)\} \tag{3.7}$$

and

$$H(f) = \tfrac{1}{2}\{\Lambda(f) + \Lambda^*(-f)\} = \tfrac{1}{2}\{H_c(f-f_o) + H_c^*(-f-f_o)\} \tag{3.8}$$

where f_o is the center frequency of the bandpass filter. For a nonsymmetrical bandpass filter with real transfer function these relations have been depicted in Figure 5.6.

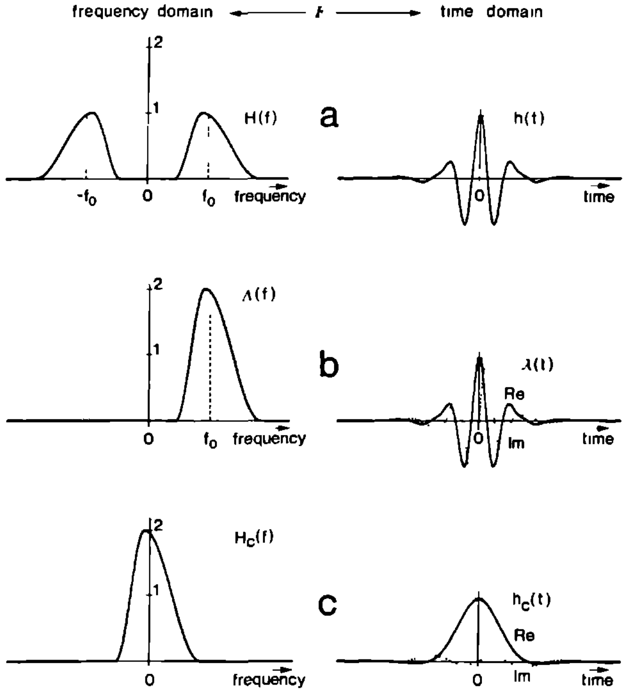

Figure 5.6a–c

Steps in obtaining the analytic impulse response and the complex envelope. a: *A (nonsymmetrical) bandpass filter with real transfer function and the corresponding filter's impulse response.* b: *Omitting the negative frequencies, doubling the magnitude of the positive ones and inverse Fourier transformation gives the analytic impulse response. The equivalent time domain operation corresponds to the addition of an imaginary component, which is the Hilbert transform of the original real impulse response.* c: *Frequency translated version of the one-sided spectrum of* b. *Its inverse Fourier transform represents the complex envelope, the modulus of which is the temporal envelope of the original impulse response of* a. *For arbitrary, real, signals the analytic signal and complex envelope can be derived in a completely analogous way*

The filter with impulse response $h_c(t)$ and transfer function $H_c(f)$ equals, except for an amplitude factor, the *equivalent lowpass filter* introduced by Papoulis (1962). The reason for this terminology becomes obvious from (3.8) and Figure 5.6, where it can be seen that $H_c(f)$ is obtained by translating the original transfer function H(f) to zero frequency, whereby its magnitude doubles [1].

Conversely, an arbitrary bandpass filter may be formed by translating the equivalent lowpass filter transfer function by an amount equal to the center frequency, according to (3.8). In the time domain, this operation corresponds to a modulation of the equivalent lowpass filter impulse response $h_c(t)$. In general, this impulse response will be a *complex* function of time. For the impulse response h(t) of the original bandpass filter, we thus have from (3.7)

$$h(t) = \mathrm{Re}\{h_c(t)\} \cos 2\pi f_o t - \mathrm{Im}\{h_c(t)\} \sin 2\pi f_o t \tag{3.9}$$

If the bandpass filter has a *real* transfer function, the function $H_c(f)$ is also real, hence $h_c(t)$ satisfies the relation $h_c(-t) = h_c^*(t)$. From (3.9) it then follows that h(t) is a real *and even* function of time. If the bandpass filter is moreover symmetrical, $H_c(f)$ is a real and even function. In that case $h_c(t)$ is real and even as well, so that the last term in (3.9) equals zero.

For non-real-time spectral analysis it is quite natural to use bandpass filters having *real* transfer functions. Such a choice implies that the filters have *zero-phase*, so that *no delay* is introduced in the bandpass filters' outputs. In the remainder of this paper we therefore restrict ourselves to purely real transfer functions.

We will now return to the determination of the temporal and spectral moments, as defined in (3.1) through (3.5).
From (3.7) it is obvious that

$$|\lambda(t)|^2 = |h_c(t)|^2 \tag{3.10}$$

and, since for real transfer functions $h_c(-t) = h_c^*(t)$, these are *even* functions of time. Therefore by inserting (3.10) into (3.1) through (3.3) we find

$$I = \int |h_c(t)|^2 \, dt = \int |H_c(f)|^2 \, df \tag{3.11}$$

$$\mu_t = 0 \tag{3.12}$$

and

----

[1]) This doubling, which is also maintained in the definitions in Table 5.1, is a logical consequence of interpreting the impulse response $h_c(t)$ of the equivalent lowpass filter as the complex envelope of h(t). This description enables a unified approach in terms of analytic signals.

$$\sigma_t^2 = \frac{1}{I}\int t^2 |h_c(t)|^2 \, dt \tag{3.13}$$

Furthermore we have

$$\mu_f = f_o \tag{3.14}$$

and by inserting this into (3.5), making use of (A9) it follows that

$$\sigma_f^2 = \frac{1}{I}\int f^2 |H_c(f)|^2 \, df \tag{3.15}$$

From these equations we conclude that the bandwidth of the transfer function and the time duration of the impulse response of a bandpass filter are uniquely determined through the equivalent lowpass filter, the impulse response of which is essentially the complex envelope of the original filter's impulse response. This implies, in particular, that the time-bandwidth product of a bandpass filter is independent of its center frequency. This result will be used in the next subsection, where time-bandwidth products of various bandpass filter types will be evaluated.

3.3 RMS Time-bandwidth Products of Some Bandpass Filters

Before discussing in more detail some of the filter types which have been investigated, we will go back to our starting point and summarize which requirements the bandpass filters should meet. We recall that the filters form part of a *bank* of (proportional bandwidth) filters, aimed at measuring the spectro-temporal energy distribution, and to be realized on a digital computer. We therefore require that:

1. The sum of the transfer functions of all filters in the bank be unity;
2. The filters have both adequate spectral *and* temporal properties and close to minimum time-bandwidth products;
3. The transfer functions be purely real (see the previous subsection), and
4. These functions be easily implementable in software.

With these requirements in mind, some well-known symmetrical filter types, along with two nonsymmetrical variants have been investigated. On the basis of the results from the previous subsection, this has been done by starting from the equivalent lowpass filter transfer functions and their calculated impulse responses, as summarized in Table 5.1. These functions have been plotted in Figure 5.7. The rms bandwidth of each filter type has been evaluated from (3.15) and (3.11). In principle, the rms duration of the impulse response can be calculated from (3.13). However, for some filters, particularly the cosine type, the expressions for the impulse response are rather complicated, so that

Filter type (symmetrical)	equivalent low pass filter transfer function H(f)		equivalent low pass filter impulse response h(t)	See Fig 5 7
rectangular (uniform)	2	$\lvert f\rvert < f_c$	$4f_c\left(\frac{\sin 2\pi f_c t}{2\pi f_c t}\right)$	a
	0	$\lvert f\rvert > f_c$		
triangular (Bartlett)	$2\left(1-\frac{\lvert f\rvert}{f_c}\right)$	$f < f_c$	$2f_c\left(\frac{\sin \pi f_c t}{\pi f_c t}\right)^2$	b
	0	$f > f_c$		
cosine (Tukey)	$1+\cos\frac{\pi f}{f_c}$	$f < f_c$	$2f_c\left(\frac{\sin 2\pi f_c t}{2\pi f_c t}\right)\left(\frac{1}{1-(2f_c t)^2}\right)$	c
	0	$f > f_c$		
exponential (Gaussian)	$2\exp\left[-\pi\left(\frac{f}{f_c}\right)^2\right]$	$-\infty < f < \infty$	$2f_c\exp[-\pi(f_c t)^2]$	d
Filter type (nonsymmetrical)				
"triangular"	$2\left[1+\frac{3\left(f+\frac{1}{6}f_c\right)}{2f_c}\right]$	$-\frac{5}{6}f_c < f < -\frac{1}{6}f_c$	$\frac{1}{\pi i t}\left[\frac{\sin\frac{4}{3}\pi f_c t}{\frac{4}{3}\pi f_c t}\exp(\pi i f_c t)\right. +$	e
	$2\left[1-\frac{3\left(f+\frac{1}{6}f_c\right)}{4f_c}\right]$	$-\frac{1}{6}f_c < f < \frac{7}{6}f_c$	$\left. - \frac{\sin\frac{2}{3}\pi f_c t}{\frac{2}{3}\pi f_c t}\exp(-\pi i f_c t)\right]$	
	0	otherwise		
"cosine"	$1+\cos\frac{3\pi(f+\nu)}{2f_c}$	$-\frac{2}{3}f_c-\nu \le f < -\nu$	$\frac{\exp(-2\pi i\nu t)}{2\pi i t}\left[\frac{\exp(\frac{8}{3}\pi i f_c t)+(\frac{8}{3}f_c t)^2}{1-(\frac{8}{3}f_c t)^2}\right. +$	f
	$1+\cos\frac{3\pi(f+\nu)}{4f_c}$	$-\nu < f < \frac{4}{3}f_c-\nu$	$\left. - \frac{\exp(-\frac{4}{3}\pi i f_c t)+(\frac{4}{3}f_c t)^2}{1-(\frac{4}{3}f_c t)^2}\right]$	
	0	otherwise		
		$\left[\nu = \frac{f_c}{3}\left(1-\frac{16}{3\pi^2}\right)\right]$		

Table 5.1 Equivalent lowpass filter transfer functions and impulse responses for various filter types

Filter type (symmetrical)	rms bandwidth (σ_f)	rms duration (σ_t)	bandwidth duration product $4\pi\sigma_f\sigma_t$	approx $4\pi\sigma_f\sigma_t$	practical BD product
rectangular (uniform)	$\frac{f_c}{\sqrt{3}}$	∞	∞		∞
triangular (Bartlett)	$\frac{f_c}{\sqrt{10}}$	$\frac{\sqrt{3}}{2\pi f_c}$	$\sqrt{\frac{6}{5}}$	1.095	3.464
cosine (Tukey)	$f_c\sqrt{\frac{1}{3}\left(1-\frac{15}{2\pi^2}\right)}$	$\frac{1}{2f_c\sqrt{3}}$	$\frac{1}{3}\sqrt{4\pi^2-30}$	1.026	≈ 3.628
exponential (Gaussian)	$\frac{f_c}{2\sqrt{\pi}}$	$\frac{1}{2f_c\sqrt{\pi}}$	1	1	
Filter type (nonsymmetrical)					
"triangular"	$\frac{f_c}{6}\sqrt{\frac{19}{5}}$	$\frac{3}{4\pi f_c}\sqrt{\frac{3}{2}}$	$\frac{1}{2}\sqrt{\frac{57}{10}}$	1.194	3.674
"cosine"	$\frac{2}{3}f_c\sqrt{\frac{3}{4}-\frac{29}{6\pi^2}-\frac{64}{9\pi^4}}$	$\frac{1}{4f_c}\sqrt{\frac{3}{2}}$	$\sqrt{\frac{1}{2}\pi^2-\frac{29}{9}-\frac{128}{27\pi^2}}$	1.110	3.848

Table 5.2 Characteristic bandwidth, duration and combined bandwidth duration measures for the equivalent lowpass filters of Table 5.1

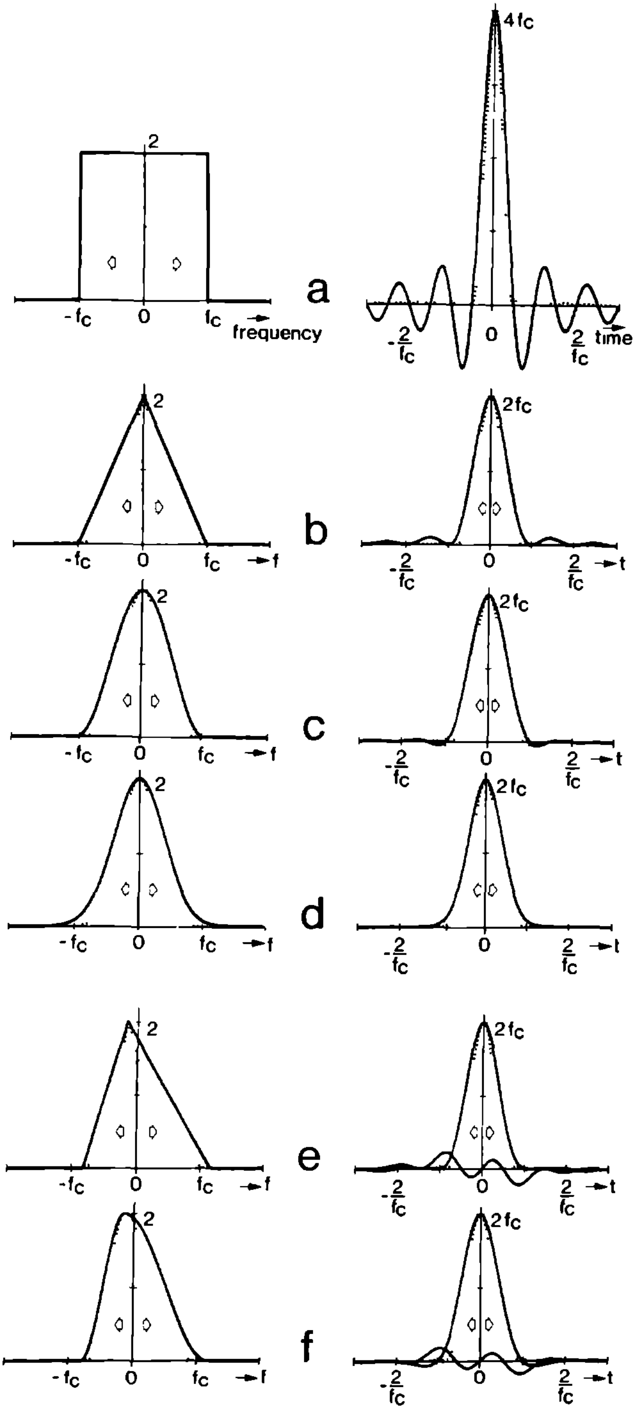

Figure 5.7a-f
Equivalent lowpass filter transfer functions (left hand side) and their calculated impulse responses (right hand side) for the various filter types summarized in Table 5.1. The dotted curves represent (arbitrarily scaled) squared magnitudes of the transfer functions and impulse responses, from which rms bandwidths and rms durations have been calculated. Actual values for the latter quantities are indicated by open arrows

direct integration according to (3.13) is not feasible. Fortunately, an elegant alternative way of calculating the rms duration is by using the transform relation

$$\sigma_t^2 = \frac{1}{I}\int t^2 |h_c(t)|^2 \, dt = \frac{1}{4\pi^2 I}\int \left|\frac{dH_c(f)}{df}\right|^2 df \qquad (3.16)$$

which is based on well-known properties of the Fourier transform. Since the filter types considered all have elementary transfer functions, the last expression of (3.16) is in all cases easily evaluated. Finally, the rms time-bandwidth product has been calculated, from which figure it can be concluded how much a particular filter type deviates from the "ideal" filter, having Gaussian envelopes in time and frequency (cf. (3.6)).

The above quantities have been summarized in Table 5.2. Inspection of this table leads to the following conclusions. The rectangular filter was naturally investigated because of its extreme simplicity (cf. the above requirements *1*,*3* and *4*). This type of filter is usually referred to as "ideal lowpass" (see e.g. Papoulis, 1962). Obviously the adjective "ideal" is restricted to properties in the frequency domain, since the duration of the impulse response in the time domain is *not finite*. This is a consequence of the impulse response damping out only very slowly. In view of our requirement *2* this type of filter is thus inadequate. The triangular and cosine filters appear to have better temporal properties. These two filters are well-known as "lag windows" in spectral smoothing (see e.g. Jenkins and Watts, 1968) and are often referred to as Bartlett and Tukey window respectively. The cosine filter, in particular, deviates less than 3% from the minimum time-bandwidth product attained with the Gaussian filter, which has also been included in the table for comparison. In view of our requirements *1* and *4* the Gausssian filter is not well suited. Theoretically, its transfer function extends over the entire frequency axis and thus would have to be truncated in practice. Moreover, the restriction that the sum of all filter transfer functions should be unity gives rise to difficulties with this filter, in contrast to the triangular and cosine types.

If a bank of proportional, or *constant relative bandwidth*, filters is required, individual transfer functions become nonsymmetrical. This is easily understood when it is realized that such filters become symmetrical if projected on a *logarithmic* frequency axis. For instance, in an octave system (Figure 5.8) the leading slope of the transfer function is twice as steep as that of the trailing slope. Therefore, in addition two nonsymmetrical filters, derived from the triangular and cosine type, have also been investigated.

The asymmetry in transfer function leads to an impulse response which is a complex function of time (see section 3.2). Also, the mean frequency f_o no longer simply coincides with the center of the passband. In this case the equivalent lowpass filter has been defined such that the mean frequency of its squared magnitude $\int f|H_c(f)|^2\, df$ equals zero, just as in the symmetrical case, where this condition is self evident. Since the transfer functions of the nonsymmetrical filters deviate still more from the "optimal" Gaussian shape it is to be expected that the time-bandwidth products increase accordingly. Nevertheless it appears that the actual figure for the cosine filter is still very acceptable. Of course, filters having more complicated transfer functions may actually lead to still lower rms time-bandwidth products. But in our opinion the cosine filter, and its nonsymmetrical counterpart, provide a fairly good compromise between performance and simplicity, and may therefore be recommended in short-time spectral analysis.

3.4 *Practical Time-bandwidth Products*

In the previous section we calculated *rms* bandwidths and durations which permitted a comparison and selection of proper filter characteristics. In practice the question arises as to what the temporal resolution at the filter's output or, alternatively, the *effective* duration of the filter's impulse response, is, once the bandwidth and the type of filter are specified. To answer this question, we have computed a "practical" time-bandwidth product for the triangular and cosine filters and their nonsymmetrical counterparts. This product is obtained in the following way. Departing from a defined transfer function in the frequency domain, the factual *bandwidth* is the interval where this transfer function differs from zero. Denoting this measure by B, we have, according to Table 5.1: $B = 2f_c$, independent of the shape of the transfer function. (The relation between B and the *rms* bandwidth σ_f does depend, however, on the filter type (cf. Table 5.2).)

From the fact that the transfer function is strictly limited in frequency it follows that the impulse response cannot be limited in time, at least theoretically. However, if we ask for an effective time duration measure, we essentially ask how the impulse response should be truncated such that most of its energy is embraced. Rihaczek (1969) suggested taking the quantity $2\pi\sigma_t$ as "effective signal duration", and this seems a reasonable, though somewhat conservative, choice. It implies a truncation at approximately $3\sigma_t$ about both the positive and negative time axis (cf. Figure 5.7), so that almost all energy is embraced. Denoting this measure by D ($= 2\pi\sigma_t$), *practical* time-bandwidth products BD are shown in the last column of Table 5.2 [1]. From these figures it follows, for example, that a factual bandwidth of 4 Hz results in a temporal resolution of slightly less than one second.

3.5 Discrete Systems

Thusfar we have analyzed continuous transfer functions and impulse responses. We will now briefly derive some of the previous relations for the discrete case. We assume that the transfer function H(f) has been adequately sampled and is represented by a sequence of K samples, (i.e. K/2 samples corresponding with positive, and an equal number corresponding with negative frequencies). Its inverse discrete Fourier transform *(DFT)* also contains K samples, representing the sampled impulse response h(t) over an assumed time interval $0 \leq t \leq T$. Note that the discrete analytic impulse response $\lambda(t)$ is simply obtained by setting K/2 "negative irequency" samples to zero, doubling the magnitude of the remaining ones and taking the inverse DFT. Similarly, the complex impulse response $h_c(t)$ is obtained by shifting the positive frequency part of the discrete transfer function to zero, doubling its magnitude, and taking the inverse DFT. Denoting the time resolution by Δt and the frequency resolution by Δf we have $\Delta t = T/K$ and $\Delta f = 1/T$, so that

$$\Delta t \Delta f = 1/K \tag{3.17}$$

If we further assume that n_f and n_t [2] represent the number of samples corresponding with the *rms* bandwidth and time duration, i.e. $\sigma_f = n_f \Delta f$ and $\sigma_t = n_t \Delta t$, it follows that

$$n_f n_t = \frac{\sigma_f \sigma_t}{\Delta f \Delta t} \geq \frac{K}{4\pi} \tag{3.18}$$

Equation (3.18) can be regarded as the discrete analogue of Gabor's uncertainty relation (3.6). Although this result may seem strange at first glance,

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[1]) The fact that the BD product is larger for the cosine than for the triangular filter is explained by our bandwidth definition, which does not take the shape of the transfer function into account. With B fixed, the rms bandwidth of the cosine filter is substantially smaller than that of the triangular filter. This explains the seeming inconsistency between the rms and the practical time-bandwidth product.

[2]) Here it has been tacitly assumed that these are natural numbers. In practice, the nearest higher integer may be inserted.

(3.18) implies that the underlying time-bandwidth product is independent of the sampling scheme. Increasing the sequence length K by increasing the sampling frequency increases n_t accordingly, while increasing the total observation time increases n_f accordingly. Thus the time-bandwidth product $n_f n_t$ of a discrete filter can be found through multiplication of the $\sigma_f \sigma_t$ product of the continuous counterpart by the sequence length K.

The same reasoning holds true for the practical time-bandwidth product BD as derived in the previous subsection. If m_f [1] corresponds with the bandwidth B and m_t [1] with the effective impulse response duration D then

$$m_f m_t = K\ BD \tag{3.19}$$

The instantaneous power of a filter's output is optimally sampled if the time interval between successive samples equals the effective duration of the impulse response. That is, over the period T, the number of samples equals

$$\frac{T}{m_t \Delta t} = \frac{T\ m_f}{K(BD)\Delta t} = \frac{m_f}{BD} \tag{3.20}$$

where use has been made of (3.19). In passing, it may be noted that (3.20) bounds m_f to a minimum value, namely $m_f \geqq BD$. Lower values for m_f lead to aliasing in the time domain, since, by (3.19) this would imply $m_t > K$, which means that the effective impulse response duration would be larger than the observation interval. In view of the figures in Table 5.2 it follows that m_f should be at least 4.

In the present context it may also be noteworthy that, once a discrete time sequence has been obtained, the total observation time and sampling frequency that were actually used are of no more importance for further processing. In particular, since the bank of bandpass filters employed in our digital short-time spectral analysis is defined in terms of sequence indices, rather than absolute frequencies, that bank may be equally applied to either a brainstem evoked potential (observation time 10 ms) or to a visual or somatosensory evoked potential (observation time 500 ms), without alteration. This fact, of course, greatly simplifies computational procedures in practice (see section 4).

Finally, the method of short-time spectral analysis as described in section 2.2.2 (Figure 5.4) implies that from the total sequence with length K, sequences of only m_f samples wide (corresponding to the bandwidths B of

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[1]) See footnote 2 on previous page

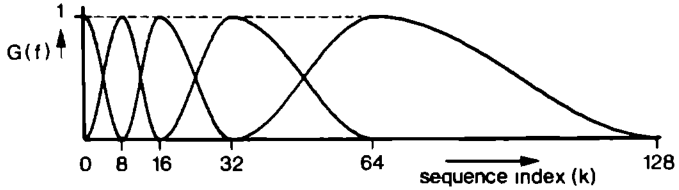


Figure 5.8

Bank of octave filters, suggested for use in digital short-time spectral analysis of evoked potential data. (Only five filters, corresponding to an original sampled sequence of 256 points, have been shown)

subsequent filters) are inverse Fourier transformed. Consequently, the time domain filter output and its instantaneous power are also described by m_f samples. In view of (3.20) it is thus concluded that in this case the factual temporal resolution is smaller by a factor BD.

4. Results and Conclusions

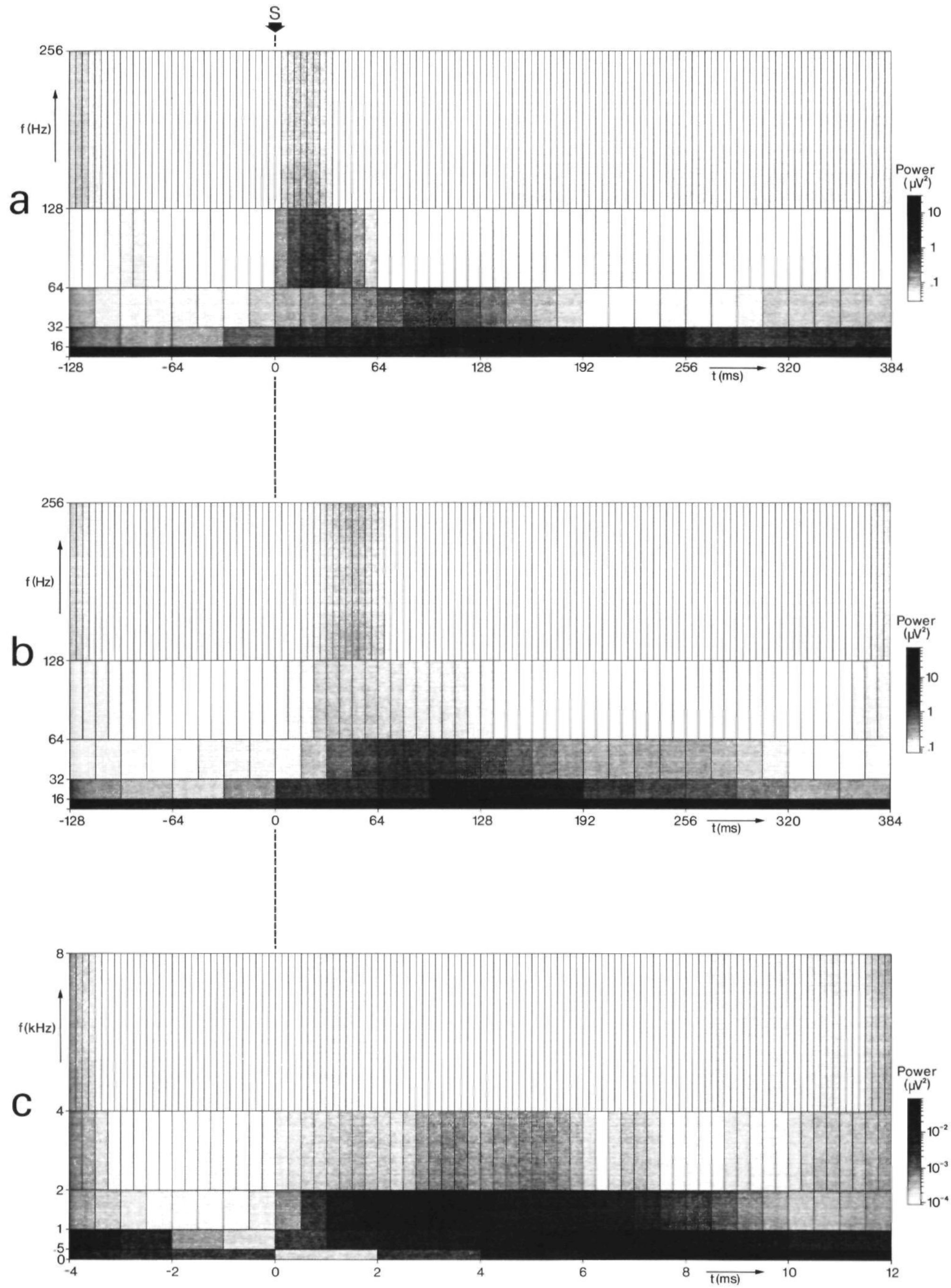
From the theoretical considerations of the previous sections we conclude that an appropriate spectro-temporal representation of an evoked potential waveform can be obtained by using a bank of constant relative bandwidth filters, whose transfer functions are of a (nonsymmetrical) cosine shape. The spectro-temporal power is obtained by computing the squared magnitude of the complex envelopes of the individual filter outputs (section 2.2.2 and Figure 5.4). As explained in section 2.3, the choice of constant relative bandwidth filters implies for the low frequency region a small bandwidth and thus good spectral resolution but, consequently, poor temporal resolution, while in the high frequency region the situation is reversed: poor spectral but good temporal resolution. In section 3.2 it was shown that a given filter type has a constant time-bandwidth product. Thus the proposed spectro-temporal representation divides the time-frequency plane into rectangles of equal area (of a size corresponding to the time-bandwidth product). The dimensions along the time and frequency axes vary inversely to each other, as a function of the center frequencies of the filters. It will presently appear that the above interpretation is quite useful in the visualization of time-varying power spectra.

Figure 5.8 shows the *octave* filterbank that we normally use in practice. As stated in section 2.3, this bank provides, in general, a reasonable compromise between the desired time and frequency resolution. The number of filters *(NF)* in this bank is given by $NF={}^2\log(K/8)$, where K is the total number of points by which the evoked potential was sampled. There is no fundamental law behind this equation. It is simply the consequence of choosing the first filter in the bank to be 8 sample points wide. The filter covers positive frequencies only, so that the total sequence length, including negative frequencies, corresponding to that filter is 16. Each doubling of this number adds an additional filter to the bank, as is obvious from figure 5.8.

Using the octave bank, time-varying power spectra were calculated for the somatosensory, visual and brainstem auditory evoked potentials of Figure 5.5. For reasons of surveyability only 256 samples (corresponding to 5 filters) were used. Figure 5.9 shows the resulting spectro-temporal power distributions, in the form of grey scale representations in the time-frequency plane. Although this form of presentation is not a compact one, it has deliberately been chosen in order to visualize the foregoing theoretical considerations. Each individual rectangle in the plane corresponds to *one sample* of the spectro-temporal description. (In principle, this spectro-temporal power distribution is described by as many samples as there are in the original time signal, i.e. 256 in this case). The size of the rectangles along the frequency axis is directly proportional to the bandwidth of successive filters. Following section 3.5 and according to earlier theoretical considerations, their size along the time axis is inversely proportional to that bandwidth.

In order to reduce the variance of the power estimates, caused by noise that remained beyond the averaging procedure, an identical smoothing in all frequency bands was performed, using a 3 point "Hanning" window. In general, however, smoothing need not be the same in all passbands, and usually higher bands require heavier smoothing due to the larger variability in these bands (see de Weerd and Kap, 1981 for further discussion).

Upon inspection of Figure 5.9, several characteristic features of the individual evoked potentials may be recognized. The somatosensory evoked potential (Figure 5.9a) shows, just after the moment of stimulation, a short-term power increase in the highest passbands, which is continued in the lower passbands at somewhat later instants in time. Also in these lower bands the power increase is of longer duration (cf.section 2.3). The power distribution of the visual evoked potential (Figure 5.9b) exhibits an essentially similar, though less pronounced, picture. In the higher passbands, the power increase occurs with longer lag time (relative to the stimulus) and lower relative intensity as compared to the somatosensory evoked potential. Obviously, these areas of increased power are caused by the "early oscillatory potentials" (Cracco and Cracco, 1978), which, upon a closer inspection, are visible in the time signal


S
a
f (Hz)
256
128
64
32
16
-128
-64
0
64
128
192
256
t (ms)
320
384
Power
(μV²)
10
1
.1
b
f (Hz)
256
128
64
32
16
-128
-64
0
64
128
192
256
t (ms)
320
384
Power
(μV²)
10
1
.1
c
f (kHz)
8
4
2
1
.5
0
-4
-2
0
2
4
6
8
t (ms)
10
12
Power
(μV²)
10⁻²
10⁻³
10⁻⁴

as well (cf. Figure 5.5b). The brainstem auditory evoked potential (Figure 5.9c) shows relatively large power concentration in the middle frequency bands. Also, this power extends over longer time intervals or, in other words, is less time-varying as compared to the distributions of the former two evoked potentials. In interpreting this observation, the totally different time and frequency scales must, of course, be kept in mind.

Although in this study we have concentrated on cerebral evoked potentials, it should be noted that the present method of short-time spectral analysis, has a much wider scope of application in electrophysiological data processing. For instance, it has been successfully applied to the analysis of peripheral compound nerve action potentials. It may also be used to advantage in studies concerning the dynamic characteristics of the spontaneous EEG, a field which has received increasing attention in recent years.

Acknowledgement

Many of the ideas, presented in this paper, have been significantly influenced by the work of P.I.M. Johannesma and the Neurophysiological Research Group from the Laboratory of Medical Physics and Biophysics, University of Nijmegen. Preliminary work on the spectro-temporal representation of evoked potentials was carried out by W.L.J. Martens, in partial fulfillment of his M.S. degree in electrical engineering, in 1977. Also, the authors are indebted to A.J.H. Vendrik and G.J.H. Uyen for their critical reading of the manuscript, and to J. Moleman for his careful preparation of Figures 5.2 and 5.9.

Figure 5.9a-c

*Grey scale representations of the spectro-temporal power distribution (time-varying power spectrum) of the evoked potentials of Figure 5.5 (*a: *somatosensory;* b: *visual;* c: *brainstem auditory evoked potential). These representations were obtained with the method of bandpass filtering, using the bank of octave filters shown in Figure 5.8. The vertical frequency scale corresponds to upper cut-off frequencies of successive filters. S: moment of stimulation. All rectangles in the time-frequency plane are of equal area, each corresponding in fact, to just* one *sample value of the time-varying power spectrum*

Appendix: Analytic Signals, the Hilbert Transform and Narrow Band Representation of Signals[1]

A *real* signal x(t) with Fourier transform X(f) contains both positive and negative frequency components, related as $X(-f) = X^*(f)$, where the starring denotes the complex conjugate. We define a new signal $\xi(t)$ with Fourier transform $\Xi(f)$:

$$\Xi(f) = \begin{cases} 2X(f) & f > 0 \\ 0 & f < 0 \end{cases} \tag{A1}$$

The new signal $\xi(t)$ is called the *analytic signal* (Gabor, 1946) or the pre-envelope (Dugundji, 1958) of x(t). Its Fourier transform contains only positive frequency components, hence $\xi(t)$ is a *complex* function of time.

An equivalent definition of the analytic signal in the time domain is given by

$$\xi(t) = x(t) + i\check{x}(t) \tag{A2}$$

where $\check{x}(t)$ is the *Hilbert transform* of the signal x(t), and $i = \sqrt{-1}$. The Fourier transform of $\check{x}(t)$ is given by

$$\check{X}(f) = -\,iX(f)\,\mathrm{sgn}(f) \tag{A3}$$

with the signum function defined as

$$\mathrm{sgn}(f) = \begin{cases} 1 & f > 0 \\ 0 & f = 0 \\ -1 & f < 0 \end{cases} \tag{A4}$$

From (A3) and (A4) it follows that the Hilbert transform can be interpreted as a 90^o phase shifter. In modulation theory the signal $\check{x}(t)$ is therefore termed the *quadrature signal* of x(t). From (A2) it is also clear that the analytic signal uniquely determines the original real signal x(t). These signals are simply related by

$$x(t) = \mathrm{Re}\{\xi(t)\} \tag{A5}$$

where Re{ } means "real part of".

In describing bandpass signals (i.e. signals whose Fourier transform is non-zero only in a limited frequency interval, not including zero frequency) it is convenient to adopt the following notation:

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1) Introductory texts on these fundamental concepts can be found in several textbooks, e.g. Sakrison (1968), Rihaczek (1969), and Whalen (1971). See also Aertsen and Johannesma (1980) for a similar treatise on this subject.

$$x(t) = \mathrm{Re}\{\xi(t)\} = \mathrm{Re}\{a(t)\exp(2\pi i f_o t)\} \tag{A6}$$

where $\xi(t)$ is again the analytic signal of $x(t)$; $a(t)$ is the *complex envelope* of $x(t)$ and f_o represents an arbitrary frequency (usually, though not necessarily, within the passband of the signal). The terminology "complex envelope" and its physical significance become obvious when we split $a(t)$ into its modulus and phase

$$a(t) = |a(t)|\exp(i\phi(t)) \tag{A7}$$

so that we can rewrite (A6) as

$$x(t) = \mathrm{Re}\{|a(t)|\exp(2\pi i f_o t + i\phi(t))\} = |a(t)|\cos(2\pi f_o t + \phi(t)) \tag{A8}$$

where $|a(t)|$ is easily recognized as the *envelope* of the modulated signal $x(t)$. Accordingly, $|a(t)|$ is referred to as the *temporal envelope* or *instantaneous amplitude* and its squared value $|a(t)|^2$ as the *temporal energy density* or *instantaneous power* of $x(t)$.

The notation (A6) is commonly termed the *narrow band representation* of signals. However, the above relations apply to arbitrary band limited signals as well (see Rihaczek (1969) for a detailed discussion on this point). Other simple and unambiguous definitions, in particular for the instantaneous phase and frequency, can be derived from (A6), but will not be further discussed here (cf. Aertsen and Johannesma, 1980).

It is easily demonstrated that the complex envelope $a(t)$ is a "lowpass version" of the analytic signal $\xi(t)$. Notably it follows from (A6) that

$$A(f) = \Xi(f+f_o) \tag{A9}$$

where $A(f)$ is the Fourier transform of $a(t)$. This relation implies that $a(t)$, unlike $\xi(t)$, is in general not an analytic signal, since $A(f)$ may contain negative frequencies (see e.g. Figure 5.6). Equation (A9) suggests a particularly useful computational method for determining the instantaneous amplitude, or power, of a bandpass signal, namely

$$|a(t)|^2 = |F^{-1}\{\Xi(f+f_o)\}|^2 \tag{A10}$$

where F^{-1} denotes the inverse Fourier transform. That is, the instantaneous power of a bandpass signal is obtained by taking its Fourier transform, omitting the negative frequencies and doubling the magnitude of the positive ones, translating this spectrum to zero frequency and applying the inverse Fourier transform, finally squaring the magnitude of the resulting complex signal (see also section 2.2.3 and Figure 5.4).

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Chapter 6

A Posteriori Time-varying Filtering of Averaged Evoked Potentials
I. Introduction and Conceptual Basis

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Biol. Cybernetics, in press (1981)

Abstract
This paper forms a preface and introduction to a new method for the estimation of evoked potentials: *a posteriori time-varying filtering*. A simple evoked potential model, consisting of a transient signal and additive noise, is discussed and the underlying assumptions explicitly formulated. Assuming this model, the problem of estimating the signal from an ensemble is considered from the statistical and communication engineering point of view, along with a brief survey of the pertinent literature. It is explained why ensemble averaging, in general, does *not* provide the best estimate in the mean-square error sense. After a summary of the controversial aspects of time-*in*variant "*a posteriori* 'Wiener' filtering", it is indicated how that method can be generalized to a time-varying counterpart, which is able to handle the essentially transient character of evoked potential waveforms. Finally, the new method is presented on a conceptual level and its application illustrated by examples.

1. Introduction
The estimation of transient evoked potentials from the human brain by means of ensemble averaging has been established clinical practice for some time. To a large extent, this is certainly due to the favourable properties of the averaging method. The technique is easily instrumented, simple to use and the interpretation of the results is intuitively easy and does not require an extensive background in signal analysis. However, one of the most serious drawbacks is that often a large number of stimulus repetitions must be presented before a reliable estimate can be obtained. This fact presents basic experimental difficulties since evoked potentials, especially the ones of cortical origin, are influenced by many factors such as changing experimental conditions, habituation processes, arousal level etc. Such influences are difficult to control in the long term and therefore set bounds to the duration of the recording session and consequently to the number of admissible stimulus repetitions.

These inherent measurement problems have inspired many investigators to search for methods which would provide a "better" estimate of the evoked potential at a substantially smaller number of stimulus repetitions as compared to conventional averaging. Several methods have been suggested to improve the efficiency and/or obtain a further improvement beyond averaging, such as "cross-correlation averaging" (Woody, 1967; Wastell, 1977), "*a posteriori* 'Wiener' filtering" (Walter, 1969; Doyle, 1975), "selective averaging" according to Pfurtscheller and Cooper (1975), "selective averaging" according to Basar et al. (1975), and "latency corrected averaging" (McGillem and Aunon, 1977; Aunon and Sencaj, 1978). Of these methods *a posteriori* "Wiener" filtering has received considerable attention. In the seventies this technique led to many controversies and to a lively discussion, mainly in the journals of *Biological Cybernetics* and *Electroencephalography and Clinical Neurophysiology*. This discussion even led to the fundamental question (Nagelkerke and Strackee, 1979) of whether it is at all possible to improve beyond ensemble averaging, a problem that we will also deal with in the present paper. Other more recent work from several authors (in particular Doyle, 1975; Ungan and Basar, 1976; Strackee and Cerri, 1977, and de Weerd and Martens, 1978) has further shown that the originally formulated *a posteriori* "Wiener" filtering has some remarkable pitfalls. Although several improvements have been suggested, a fundamental problem which remains is that "Wiener" filtering is essentially a time-*in*variant filtering, which cannot be expected to yield optimal results when dealing with time-varying, *transient*, signals typical of evoked potential waveforms. On the other hand we are faced with apparently succesful applications of "Wiener" filtering in situations where the signal is of a more periodic character (see e.g. Hartwell and Erwin, 1976). This fact clearly justifies a search, along the lines of *a posteriori* "Wiener" filtering, for a more elaborate technique which can cope with transient waveforms.

Time-varying filtering is one such technique and essentially it can be considered as a generalized "Wiener" filtering method. A short communication and some preliminary results of this technique were initially presented at the 9th International Congress of Electroencephalography and Clinical Neurophysiology in Amsterdam, 1977 (de Weerd et al., 1977). Since then, the method has been further developed and improved, and in its present form applied to a vast amount of clinically measured auditory, visual and somatosensory evoked potentials (including peripheral nerve and spinal evoked potentials).

The aim of the present paper and the companion one (de Weerd and Kap, 1981b) is to outline the conceptual and theoretical basis for these applications and to provide a practical estimation scheme which, as experience has proved, can readily be interfaced with almost any current software averaging procedure.

The present paper is primarily a preface to the time-varying filtering technique. In section 2 the signal-plus-noise model to which this technique applies is discussed in detail. Particular emphasis is given to the essentially different types of nonstationarity that can be encountered in evoked potential ensembles. Section 3 deals with the evoked potential estimation problem. Firstly this problem is considered from the statistical point of view, which leads to the conclusion that ensemble averaging does not provide the best estimate in the mean-square error sense. In a search for estimators which might provide an improved estimation over averaging, a brief survey of the literature on optimal signal estimation is presented. Finally, it is shown how the method of *a posteriori* "Wiener" filtering can be generalized to a time-varying equivalent. In section 4 the main lines of the actual filtering procedure are described and its application illustrated. In conclusion, the last section contains a brief discussion on the merits of this new technique. The companion paper describes its further elaboration, including details of the computational procedures that are involved.

2. A Simplified Evoked Potential Model Reviewed

We consider the following evoked potential model (often referred to as the *fixed latency model*):

$$x_i(t) = s(t) + n_i(t) \qquad i = 1,2, \ldots\ldots N; \qquad 0 \leqq t \leqq T \tag{2.1}$$

where $s(t)$ represents the signal (i.e. the desired "evoked potential"), $n_i(t)$ noise (i.e. unwanted activity, not systematically related to the stimulus), N the number of stimulus presentations and T the duration of the evoked potential. The collection of responses $\{x_i(t)\}$ will be referred to as the "ensemble". For a clear understanding, not only of the time-varying filtering technique to be discussed, but more generally of advanced evoked potential estimation techniques, it will be necessary to formulate explicitly the underlying assumptions regarding the signal, noise and ensemble characteristics. This will be done after a brief discussion on the merits of the proposed model.

2.1 Comments on the Model

How valid is the signal-plus-noise model given in (2.1)? Of course, this model does not pretend to be a *physical* model for the generating process of evoked potentials. It is merely a *descriptive* model for the traditional evoked potential concept, in which it is assumed that the "evoked potential" is the *consistent feature* in evoked response ensembles [1]. This assertion is perhaps more obvious if we write (2.1) in the "stochastic form":

$$x_i(t) = E\{x(t)\} + (x_i(t) - E\{x(t)\}) \tag{2.2}$$

where E{ } means *expectation* taken over the ensemble. The first term on the right hand side is readily identified as the signal; the second term as the (zero mean) noise component of (2.1). The fact that ensemble averaging actually leads to waveforms which are, to a large extent, reproducible upon repetition of the experiment apparently provides support for the general usefulness of the proposed model.

More specifically, the stable characteristics of *brainstem* evoked potentials imply that responses elicited from these structures are well described by (2.1). On the other hand it is well known that the neurophysiological process generating the *cortical* evoked activity, produces responses which may show considerable trial to trial variability and correlation to the ongoing spontaneous EEG. Cigánek (1969), for example, provided evidence that individual waveform components of single trial visual evoked potentials show a variability in latency. In such a case, ensemble averaging only provides a (first order) approximation to some "true" underlying evoked potential waveform.

In this context we should mention the *variable latency* evoked potential *model* (Woody, 1967):

$$z_i(t) = s(t-\tau_i) + n_i(t) \tag{2.3}$$

where τ_i represents a random time variable. It has been suggested that the individual τ_i can reliably be estimated by cross-correlation techniques, either with a fixed, or with an adaptive template; whereafter the ensemble average might be improved by including only selected trials in the average (Pfurtscheller and Cooper, 1975; Aunon and Sencaj, 1978), or by aligning individual trials, using iterative cross-correlation averaging procedures (Woody, 1967; Wastell, 1977; Wastell and Kleinman, 1980). By using such techniques it is thus possible to modify the ensemble $\{z_i(t)\}$ into a new ensemble $\{x_i(t)\}$ for

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[1] Here it has been assumed that no *consistent* components from other generating sources are present (cf. section 2.2).

which (2.1) will hold.

In view of these considerations, we will assume in the following that we deal with an evoked response ensemble which can be represented by the model in (2.1). That does not mean, however, that we consider a possible trial to trial variability to be insignificant. It is quite conceivable that such variability carries significant information about the underlying nervous system. The development of complementary evoked potential models and analysis techniques for measuring that variability remains therefore indicated.

2.2 Further Assumptions Regarding Signal and Noise

In the evoked potential model given in (2.1) the *signal* is considered as fixed or *deterministic*. It should be noted, however, that this model is restricted to individual recording sessions. If the model in (2.1) were to hold in a much wider sense, e.g. for the same subject across *different recording sessions*, we had to assign certain *random properties* to the signal. Actually, the signal varies in relation to a multitude of biological (alertness, habituation etc.) and other (time of day, experimental conditions etc.) influences. This problem is the main complicating factor in evoked potential estimation, because the signal can only be reasonably assumed to remain invariant over short intervals of time, so that the number of stimulus presentations is necessarily limited.

Besides being deterministic, the signal s(t) is also a *transient* signal. This is evident, not only from the typical structure of evoked potential waveforms (see de Weerd and Kap, 1981a), but also from their distinct onset and finite duration. It should be further noted that the model in (2.1), or more specifically (2.2), implies that any component which is time locked to the stimulus, is regarded as being part of the signal. This also includes *unwanted* systematic components, such as stimulus locked muscle potentials and electrical artefacts from stimulators.

For a closer inspection of the assumed properties of the *noise*, which is considered as a *random*, zero mean process, we need to distinguish between various contributing sources. The major contributor probably derives from the spontaneous activity, or background EEG. But the possible variability of the evoked potential, as discussed before, also adds to this term (cf. (2.2)). Furthermore there are the electrical activity of other biological sources (EMG, EOG, ECG), mains interference and instrumental noise that contribute, provided, of course, that this activity is not time locked to the stimulus. It appears that sometimes a short term suppression of the background activity occurs upon presentation of a stimulus (see e.g. Cigánek, 1969). This sug-

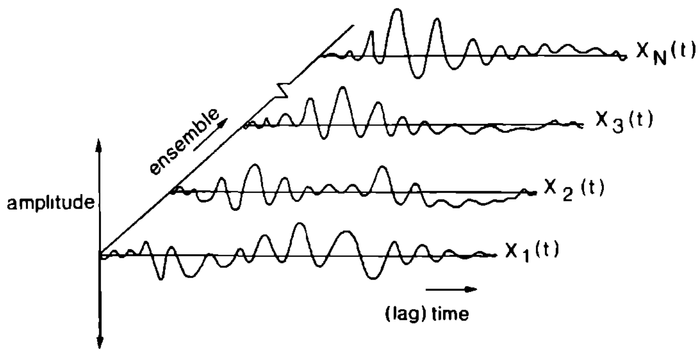

Figure 6.1

In an ensemble of evoked responses two essentially different types of nonstationarity should be distinguished, namely local *nonstationarity within single ensemble elements (associated with (lag-)time relative to the stimulus onset) and* global *nonstationarity across the entire ensemble (associated with the rank in the ensemble at fixed lag times). To avoid confusion in terminology it is proposed to indicate the latter type of nonstationarity by the term* inhomogeneity

gests that the noise may be of a *nonstationary* character. In the next subsection we will elaborate on what is meant by this latter notion.

2.3 Ensemble Properties

From the model in (2.1) it follows that the process $\{x_i(t)\}$ is a *mixed process* (Middleton, 1960), since it is essentially random, but contains in addition a deterministic component. Apart from the question of whether the noise is stationary or not, we thus conclude that the process $\{x_i(t)\}$ is *nonstationary* since its mean value, that is, the signal component (cf. (2.2)), varies with time. To avoid any misconception, it will be crucial, in particular for a full understanding of the merits of time-varying filtering, to make a clear distinction between *local* and *global* [1] nonstationarity or, in other words, between nonstationarity across *lag time* on the one hand and across the *ensemble* on the other hand (Figure 6.1):

a) Nonstationarity across lag time (local nonstationarity)

This means nonstationarity within *a single member of the ensemble*, associated with lag time, relative to the stimulus onset. As pointed out in the foregoing

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[1] The author is indebted to Th. Gasser for suggesting this terminology.

discussion, individual ensemble elements show this type of nonstationarity, due to the presence of a deterministic, transient signal component. Possible nonstationarity of the noise is a further (though probably less important) factor.

b) Nonstationarity across the ensemble (global nonstationarity)

By this we mean nonstationarity, at specified fixed lag times across the collection of ensemble elements (associated with the rank in the ensemble). This may arise due to changes in the signal waveshape and/or statistical properties of the noise *in the course of the recording session.* Notorious examples are trends due to habituation processes, changing background activity due to somnolence and changing experimental conditions. These are the types of statistical changes that are usually referred to as nonstationarities in the literature. In the foregoing discussion we have already assumed that the signal does not change within one particular recording session, and we will make the same assumption with respect to the noise, and thus to the ensemble as well.

Since the above assumptions readily lead to terminological confusion, we propose to reserve the term *nonstationarity* for (local) changes which refer to lag time relative to the stimulus and to use the new term *inhomogeneity* for (global) changes which refer to the ensemble. Thus an ensemble is homogeneous if the statistical properties of the subsequent ensemble elements do not vary with their rank in the ensemble. The ensemble would moreover be stationary when the statistical properties of the individual ensemble elements did not vary with lag time relative to the stimulus (i.e. if the signal were a *constant* (or d-c) component, and the noise stationary).

Clinical practice demonstrates that the degree of inhomogeneity is to some extent controllable (in particular by using short recording sessions, "optimal" stimulation and recording conditions, proper artefact detection etc.). In contrast, the degree of nonstationarity is an uncontrollable experimental datum. The technique of time-varying filtering has been developed to cope with this latter problem, whereas it is assumed that the global signal and noise properties remain essentially the same throughout the recording session. In other words: the underlying assumption of time-varying filtering is that we have a *homogeneous but nonstationary ensemble.*

3. Estimation of Evoked Potentials

On the basis of the assumed model of the previous section, we will now deal with the problem of optimal estimation of the evoked potential s(t) from the ensemble of responses $\{x_i(t)\}$. Thereby we make the usual choice of the least mean-square error as criterion for optimality. It appears that the question of an optimal estimator can be reformulated as the problem of finding an estimator which improves the ensemble average. In evoked potential literature, there is considerable controversy on this point. It has been argued on theoretical grounds (Strackee, 1979; Nagelkerke and Strackee, 1979) that in the absence of additional *a priori* knowledge on signal and noise no further improvement beyond ensemble averaging is possible. On the other hand, undeniable experimental evidence has been provided which supports the contrary view. For instance, using the technique of *a posteriori* "Wiener" filtering (see section 3.3) de Weerd and Martens (1978) and Kearney (1979) obtained significantly lower mean-square errors as compared to normal averaging.

We will clarify this obvious discrepancy by investigating the estimation problem more carefully from the statistical point of view (section 3.1). Next we will briefly consider the problem in a communication engineering context, which leads to the conclusion that no fully applicable theory seems to be available at present. In section 3.3 we treat the *a posteriori* "Wiener" filtering in some detail and we develop a generalization, which will form the basis for time-varying filtering. Finally, in section 3.4 we point to some of the theoretical questions that still remain to be answered.

3.1 Estimation From the Statistical Point of View

If the model in (2.1) holds, and the noise is (approximately) normally distributed [1)] the most obvious estimate to start with is the ensemble average:

$$\bar{x}(t) = s(t) + \frac{1}{N} \sum_{i=1}^{N} n_i(t) \tag{3.1}$$

or, turning to the *discrete* case:

$$\bar{x}_k = s_k + \frac{1}{N} \sum_{i=1}^{N} n_{ik} \qquad k = 1,2, \ldots\ldots\ldots K \tag{3.2}$$

If we consider each of the estimates $\bar{x}_k$ individually, then this average is, in the absence of any additional *a priori* knowledge on signal and noise [2)],

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[1)] If this were not the case, other methods of estimation, such as computation of the *median*, or *robust averaging* might prove more useful (see Gasser et al., 1981, and also Hampel, 1974 for a review).

[2)] This restriction is essential, see section 3.2.

the *optimal* estimate in the mean-square error sense. Moreover, the estimate is unbiased. In mathematical terms we thus have

$$E\{\bar{x}_k\} = s_k \qquad k = 1,2, \dots\dots\dots K \tag{3.3}$$

and

$$\varepsilon_k = E\{|\bar{x}_k - s_k|^2\} = \frac{\sigma_k^{\ 2}}{N} \qquad k = 1,2, \dots\dots\dots K \tag{3.4}$$

where $\sigma_k^{\ 2}$ is the instantaneous variance (or instantaneous noise power). The above situation is exactly the one considered by Strackee (1979) and Nagelkerke and Strackee (1979) and thus it may seem that these authors were right in arguing that no improvement beyond averaging is possible.

The crucial point, however, is that in evoked potential estimation one is not really interested in estimating *individual points*, but rather in estimating the *coherent structure* of these points, that is, the waveform of the signal. Denoting such a coherent estimator by $\hat{s}_k$ (k = 1,2, ... K), this implies that we seek to minimize:

$$\delta = E\{\sum_{k=1}^{K} |\hat{s}_k - s_k|^2\} \tag{3.5}$$

rather than the expected squared error ε_k of each of the samples individually [1)]. In other words: the estimation of evoked potentials is a *simultaneous (multidimensional) estimation problem*, rather than a number of *separate point estimation problems*. Here we have arrived at the heart of the controversy: Nagelkerke and Strackee (1979) considered the problem from the latter (one-dimensional) viewpoint, while de Weerd and Martens (1978), implicitly using the optimality criterion of (3.5), followed the opposite (multidimensional) approach.

Of course, the question remains to be answered whether minimization of (3.5) actually leads to estimators which are superior to conventional averaging. It appears that such estimators exist and, in fact, Nagelkerke (1979) came forward with some pertinent literature during one of several discussions which were held to clarify the aforementioned controversy. In particular, it was found

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1) In practice it may perhaps be more appropriate to minimize the mean-square error over *subsets* of (clearly coherent) samples, rather than minimizing this error over the *entire sequence*. However, at the present time this apparently remains a matter of intuition.

that the technique of *a posteriori* "Wiener" filtering shows a close resemblance to statistical techniques known as *"empirical Bayes estimators"*. An example of such an estimator is the "James-Stein estimator" (James and Stein, 1961). A fascinating and very readable treatise on this estimator (which in the past led to "frequent and sometimes angry debate") is given in a paper by Efron and Morris (1977). It would carry us too far to enter into the details of Stein's estimator, but the peculiar thing in it is that it provides a uniformly lower total mean-square error (cf. (3.5)) if *three or more* means are to be simultaneously estimated (i.e. $K \geq 3$). Translated to our evoked potential estimation problem, this implies that Stein's estimator would provide more accurate signal estimates, as compared to ensemble averaging, no matter what the "true" underlying signal may be. That does, however, not imply, that each individual sample point s_k is also more accurately estimated. In fact, Stein's method may occasionally lead to a worse estimate, if only an individual point is considered in isolation.

Empirical Bayes estimators all have in common the fact that the estimated points $\hat{s}_k$ are shrunk to some central value (usually the average value of all $\bar{x}_k$: $\frac{1}{K}\sum_{k=1}^{K} \bar{x}_k$, i.e. the "d-c" component) by factors which are determined through the data themselves. In the James-Stein estimator these factors are the same for all sample points $\hat{s}_k$ and therefore this estimator is not appropriate for our evoked potential estimation problem. However, much more sophisticated methods exist (see e.g. Efron and Morris, 1972; 1973), in which the estimated structure of the signal and the noise is taken into account. The problem is, however, that for practical applications these estimators cannot be derived in a straightforward mathematical manner. Their precise nature remains to some extent a matter of intuition and heuristic reasoning, sometimes leading to complicated estimation schemes (see e.g. Efron and Morris, 1973).

Another interesting connection to the actual evoked potential estimation problem can be found in a closely related field, namely that of *optimal curve estimation* (see Gasser and Rosenblatt, 1979 for a review). In that field only one particular realization of the process (2.1) is usually available and the problem is, as before, to minimize the mean-square error (3.5). Solutions of this problem are essentially based on the application of a *smoothing window*, and self-adapting algorithms, which are useful in the absence of additional knowledge on signal and noise, have been proposed as well (see e.g. Wahba and Wold, 1975). These schemes, however, lead to windows with a constant width over the entire duration of the curve. Therefore, these schemes are less suited to the smoothing of time-varying, transient signals (although they may actually lead to an improved estimation beyond averaging, in the sense of (3.5)). Papoulis (1977), who assumes a time-varying signal and additive , white noise, suggests the use of a time-varying smoothing window, in the sense that the width of that window is continuously adapted by the second derivative of the underlying signal. In this case, of course, the problem is how the instantaneous second derivative of that signal should be reliably estimated from the data themselves.

In summary we conclude that waveform estimation by means of ensemble averaging is, under the assumptions of (2.1) and (3.5), not optimal. On the other hand, statistical theory does not readily lead to practical alternatives. However, waveform estimation obviously comes within the scope of optimum prediction and filter theory, and we will follow this approach in the next subsection.

3.2 Estimation From the Engineering Point of View

In communication engineering, a frequently encountered problem is the extraction of a desired signal from a mixture of signal and noise. Thus the basic model to start with is that in (3.2) and the question is whether we can, by a linear operation on $\bar{x}_k$, estimate the signal s_k optimally in the sense of minimizing (3.5). This question leads essentially to the one posed in the previous subsection, namely whether or not it is possible to improve beyond averaging.

It appears to be a generally held belief that the estimation of a *non-random* (deterministic) signal in additive noise has been extensively studied and is dealt with in many textbooks on communication theory. Unfortunately, however, this is not the case. The *estimation problem* may easily be confused with the *detection* problem. Briefly, in the former case the signal is usually known to be present, but the problem is to recover its waveform from the masking noise. In the detection case the situation is opposite: the waveform is usually known *a priori*, but it is the presence (or absence) of this waveform which has to be estimated. Matched filters or correlation techniques are then among the well-known solutions.

In communication engineering the most celebrated estimation theory is undoubtely the "Wiener" filter theory (Wiener, 1949). Wiener's theory assumes both the signal and the noise to be random, stationary (and ergodic) processes with *a priori* known correlation functions. With these assumptions an optimal (time *in*variant) filter can be calculated. Most textbooks follow these assumptions and emphasize the problem of finding a physically realizable filter, i.e. a filter that does not respond before the input has been applied (h(t) = 0 for t < 0). Along the same lines, Lee (1960) deals with the pure filtering problem of *deterministic* signals, *in the absence of noise*. The problem here is to find a physically realizable system, using Wiener's spectral factorization theorem, to produce a desired output when the input is a specified function of time. Extensions to *time-varying systems* have been proposed (e.g. Bendat, 1958), but these are again restricted to entirely *random* (nonstationary) processes. To some extent, however, the earlier literature has dealt with the problem of estimating a nonrandom signal in additive noise. Zadeh and Ragazzini (1950) generalized Wiener's theory by assuming stationary noise and a signal consisting of two parts, the first part being a nonrandom function of time, representable by a polynomial of degree not greater than a specified number n, and the second part being stationary, random noise. This theory was later extended by Davis (1952) who permitted the random components of both signal and noise to be *non*stationary functions of time. Blum (1956) generalized the theory in an other direction, retaining the stationarity assumption, but assuming the nonrandom component of the input signal to be an arbitrary (unknown) linear function of an *a priori* known subset of analytic functions. However, the aforementioned studies are restricted to physically realizable *time-invariant* systems, and, most importantly, they all depart from an *unbiased* minimization of (3.4) rather than a minimization of (3.5). This implies that the resulting systems must necessarily rely on explicit *additional* information regarding signal and noise, since without this knowledge no further unbiased improvement is possible (cf. section 3.1). For instance, if it were known *a priori* that s(t) is band-limited to, say f_c, a lowpass filter with cutoff frequency f_c would reduce the mean-square error while the estimator remained unbiased. Similarly, if s(t) were known to be a polynomial of specified degree, (the case considered by Zadeh and Ragazzini), smoothing would also allow a bias free variance reduction. In these cases, the possibility of improving beyond averaging is self-evident.

From this brief survey, it appears that the previous studies are not applicable to our situation. We do not have *a priori* information on signal and noise; at most some of their characteristics may be estimated from the data. Furthermore, the transient structure of the deterministic signal s(t) clearly calls for the use of a *time-varying* system. And also we can relax the restriction of physical realizability , since digital signal analysis techniques will be employed. We thus conclude that from the communication engineering point of view also, no fully applicable theory seems to be available at present.

3.3 A Posteriori "Wiener" Filtering and a Generalization

Apart from the theoretical developments, as discussed in the foregoing subsections, applied papers in the evoked potential field suggested an appealing estimation method, now known as "*a posteriori* 'Wiener' filtering" (*a.p.w.f.*). It started with a paper by Walter (1969), followed by other studies suggesting modifications and improvements (Doyle, 1975; de Weerd and Martens, 1978). The basic train of thought is as follows. Departing again from the model in (2.1) we make up the ensemble average (3.1). Then, if the power density spectra of signal and noise were known *a priori*, the optimal "Wiener" filter transfer function for use on this ensemble average would read (Doyle, 1975):

$$H(f) = \frac{\Gamma_{ss}(f)}{\Gamma_{ss}(f) + \frac{1}{N}\,\Gamma_{nn}(f)} \tag{3.6}$$

where $\Gamma_{ss}(f)$ and $\Gamma_{nn}(f)$ represent known power density spectra of the signal and the noise. In practice these spectra are of course unknown, but can be estimated from the ensemble; details of this procedure are worked out in the companion paper (de Weerd and Kap, 1981b). Thus the *estimated* transfer function is given by

$$\hat{H}(f) = \frac{\Phi_{ss}(f)}{\Phi_{ss}(f) + \frac{1}{N}\,\Phi_{nn}(f)} \tag{3.7}$$

where $\Phi_{ss}(f)$ and $\Phi_{nn}(f)$ are the *estimated* spectra of the signal and the noise. In recent years this method has been subject to many controversies, which can be basically reduced to the following theoretical problems.

1. The signal s(t) is deterministic, rather than random as assumed in Wiener's original theory. However, McGillem and Aunon (1977), applying a time domain technique essentially equivalent to a.p.w.f. concluded that the formalism for deterministic signals is virtually the same as that which would be used if the signals were assumed to be random and stationary.
2. The question is whether it makes any sense to replace the suppositional *a priori* known power density spectra of signal and noise by their estimated counterparts, obtained from the data themselves. Basically, this is the same question as is posed in the theory of empirical Bayes estimators. The general conclusion is that it can be profitable to use estimators

wherein presupposed known quantities are replaced by estimated counterparts although this is at the price of a loss in optimality. In their study of *a posteriori* "Wiener" filtering de Weerd and Martens (1978) arrived at essentially the same conclusion on the basis of simulation studies.

3. Since we are dealing with a transient time-varying waveform, it is quite unlikely that *time-invariant* systems will produce optimal results. In particular, the instantaneous power of a transient signal cannot be adequately represented by a time-*in*variant descriptor such as the power density spectrum (de Weerd and Kap, 1981a). This problem can be solved by replacing the power density spectra which appear in (3.7) by their time-varying counterparts. It is this substitution which forms the key to the proposed method of *a posteriori time-varying filtering.*

Under the assumption of a homogeneous, but nonstationary, ensemble (cf. section 2.3) we thus obtain the *"optimal" time-varying system function:*

$$\hat{G}(t,f) = \frac{\Phi_{ss}(t,f)}{\Phi_{ss}(t,f) + \frac{1}{N}\,\Phi_{nn}(t,f)} \tag{3.8}$$

where t refers to lag time relative to the stimulus onset. In this case $\hat{G}(t,f)$ is no longer a transfer function in the classical sense. It is, rather, a weighing function in the combined time-frequency plane. Its interpretation and application to the ensemble average $\bar{x}(t)$ will be discussed in section 4.

3.4 Concluding Comments

Hitherto we have been unable to derive (3.8) in a rigorous mathematical form. The theoretical difficulties mainly lie in the fact that the weighing function $\hat{G}(t,f)$ is itself an estimate, rather than a known function of time and frequency. If we assume for the moment that the time-varying spectra of signal and noise were known *a priori*, it seems possible to generalize Bendat's (1958) theory of nonstationary processes to include nonrandom, time-varying, signals. However, the elaboration of this problem remains to be dealt with in the future. The actual capabilities of the proposed method have thus far been evaluated by extensive simulation studies and are further discussed in the companion paper (de Weerd and Kap, 1981b).

It is beyond the scope of this paper to establish the precise relations between the *a posteriori* filters, (3.7) and (3.8), on the one hand and empirical Bayes estimators on the other hand. The similarities, especially with more advanced empirical Bayes estimators (Efron and Morris, 1972; 1973) are striking and even include pecularities such as "clipping" of unlikely values (de Weerd and Martens, 1978). As far as we are aware these relations have not been recognized before, and their further elaboration is intended. The relevance of such a study appears to be twofold. Firstly it may provide a statistical basis for the mainly intuitive methods of *a posteriori* "Wiener" and time-varying filtering, and on the other hand it may indicate new ways of statistical estimation by considering advanced methods of optimal filtering and prediction theory.

4. Time-varying Filtering

Although it might appear from the previous section that "*a posteriori* 'Wiener' filtering" is easily generalized to a time-varying filtering method, the actual elaboration of (3.8) is not as straightforward. The problem is that the definition of a time-varying power spectrum is less obvious than it may seem at first sight. According to a fundamental principle, known as the uncertainty principle (Gabor, 1946), the power of a signal cannot be localized within an arbitrarily small region in time and *simultaneously* within an arbitrarily narrow band in frequency. Time and frequency are inversely related quantities and thus no physically meaningful power measure can provide both arbitrarily large temporal and spectral resolution at the same time. In other words, any measurement of power requires an effective bandwidth and an effective observation interval (time width). These (inversely related) spectral and temporal widths should be ideally chosen such that possible power variations within the covered time-frequency area can safely be ignored.

On the basis of these considerations, suitable ways for obtaining time-varying power spectra of evoked potentials can be established, as shown by de Weerd and Kap (1981a). Their paper leads to the conclusion that the obvious solution is the use of a bank of proportional bandwidth filters, i.e. filters having a bandwidth proportional to their center frequency. Such a filter bank implies, for the low frequency region, a small bandwidth (good spectral, but poor temporal resolution) in order to provide an accurate frequency description of the power of "slow" evoked potential components of longer duration, while for the high frequency region the bandwidth is large (poor spectral, but good temporal resolution) in order to provide an accurate time-location of the power of the "fast" components of short duration. The desired time-varying power spectrum can be obtained by squaring the outputs of the individual filters in the bank, followed by appropriate smoothing or lowpass filter-

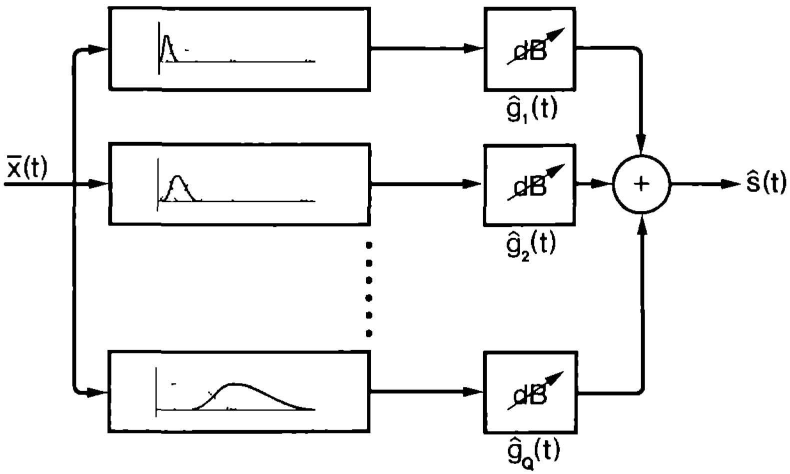


Figure 6.2
The time-varying filter, as developed in this paper, can be physically characterized as a bank of constant relative bandwidth filters, followed by time-varying attenuators and a summing network. The attenuators are controlled by the estimated time-varying signal-to-noise power ratio in the corresponding frequency bands

ing. (In the ultimate time-varying filtering method a different procedure is followed, but this is, in the present context, irrelevant.) Consequently, one obtains a set of power waveforms, each corresponding to a particular frequency band. The resulting time-varying power spectrum can therefore be written as

$$\Phi_{yy}(t,f_q) = I\{y_q(t)\} \qquad q = 1,2, \ldots\ldots\ldots Q \tag{4.1}$$

where f_q represents the center frequency of the q-th bandpass filter in a bank of Q filters, I{ } means "instantaneous (temporal) power of", and $y_q(t)$ is the output of the q-th bandpass filter when y(t) is applied as input.

Inserting (4.1) into (3.8) leads to an estimated system function

$$\hat{G}(t,f_q) = \hat{g}_q(t) \qquad q = 1,2, \ldots\ldots\ldots Q \tag{4.2}$$

with

$$\hat{g}_q(t) = \frac{I\{s_q(t)\}}{I\{s_q(t)\} + \frac{1}{N}\, I\{n_q(t)\}} \tag{4.3}$$

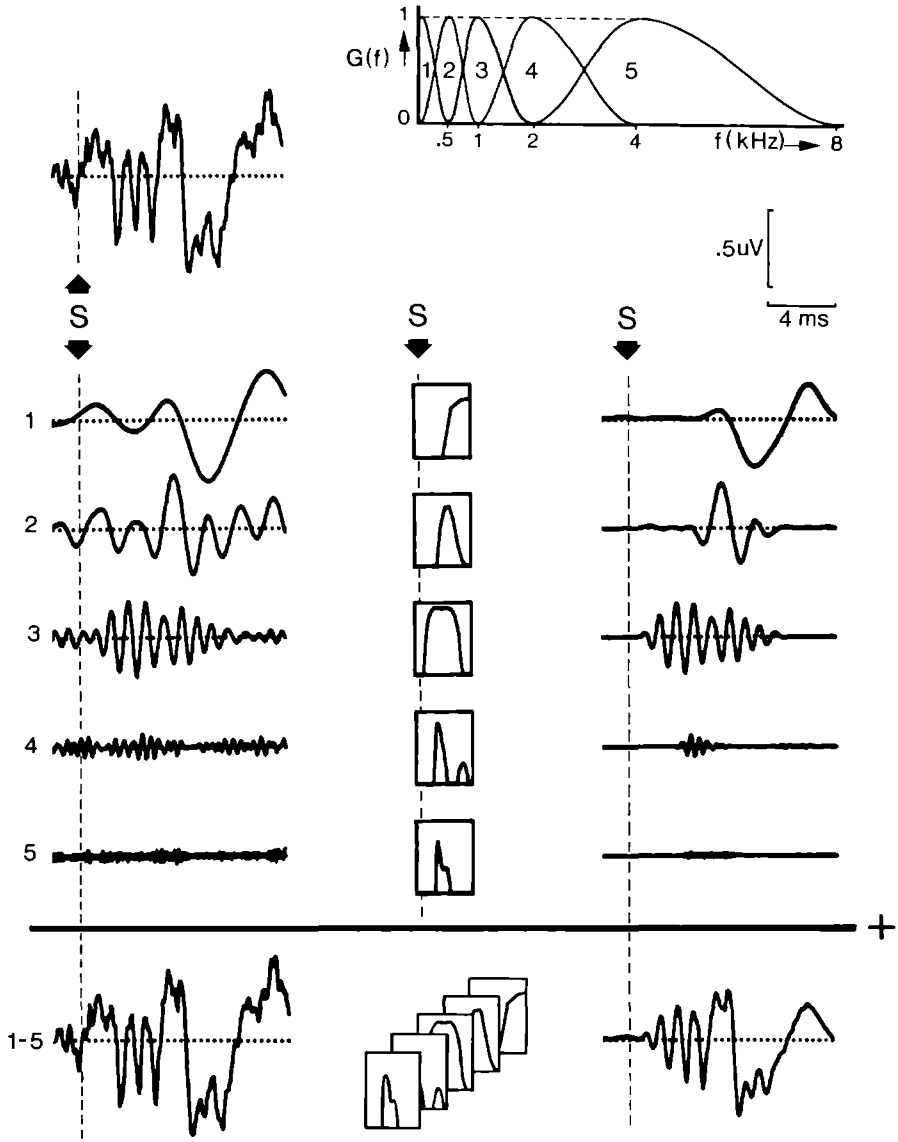
where $I\{s_q(t)\}$ and $I\{n_q(t)\}$ represent the estimated instantaneous power waveforms of the signal and the noise in the q-th passband and $\hat{g}_q(t)$ the corresponding time-varying weighing function in the same band. Theoretically, these weighing functions assume values between 0 and 1 and can therefore be interpreted as *time-varying attenuators*, which depend on the time-varying signal-to-noise power ratio in the corresponding frequency bands.

The system function $\hat{G}(t,f_q)$ which combines all passbands to cover the entire signal bandwidth, can be regarded as a weighing function in the simultaneous time-frequency domain. Its application to the ensemble average $\bar{x}(t)$ follows by passing $\bar{x}(t)$ through the same bank of bandpass filters, applying the appropriate weighing function $\hat{g}_q(t)$ to each of the bandpass filter outputs and summing the resulting waveforms over all passbands, as shown in Figure 6.2. Thus the final time-varying filtered estimate $\hat{s}(t)$ is obtained as

$$\hat{s}(t) = \sum_{q=1}^{Q} \hat{g}_q(t)\, \bar{x}_q(t) \tag{4.4}$$

where $\bar{x}_q(t)$ is the output of the q-th bandpass filter when $\bar{x}(t)$ is applied as input. It may be remarked that the application of the weighing function $\hat{g}_q(t)$ to $\bar{x}_q(t)$ in (4.4) represents a normal *multiplication* rather than a *convolution* in the *time domain*. This is in notable contrast to the a.p.w.f. technique, where filtering is performed by multiplication in the *frequency domain* and the final "Wiener" filtered estimate obtained after inverse Fourier transformation.

In principle (4.3) and (4.4) form the basis for the time-varying filtering method. However, at this point we must emphasize that the foregoing exposition has been of conceptual rather than practical utility. We have not stated how the power of the signal and noise appearing in (4.3) can be estimated from the ensemble. The actual estimation procedure leads to practical problems which make some modifications of (4.3) required. Also, we have not specified the bank of bandpass filters, nor have we discussed the actual mathematical procedures for determining the instantaneous power at the output of the filters. The latter procedures lead to refinements which enable fast and efficient schemes for the ultimate time-varying filtering method. Since, in the present paper, our aim is to explain the basic principles of the method, the aforementioned details will not be further discussed here, but will be deferred to the companion paper (de Weerd and Kap, 1981b). The overall time-varying filtering procedure, including examples of the waveforms appearing at the various stages of the processing of Figure 6.2, has been illustrated in Fig-



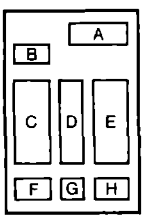


Figure 6.3A-H

Illustrative explanation of the basic principles of time-varying filtering (t.v.f.), *applied to a brainstem auditory evoked potential (average of 128 sweeps), after binaural click stimulation (70 dB SL).* S: *moment of stimulation. Using a* ure 6.3. In this figure time-varying filtering is applied to a (normal) brainstem auditory evoked potential (average of 128 sweeps), shown in Figure 6.3B. The averaged evoked potential forms the input to a bank of bandpass filters (Figure 6.3A), whose output waveforms are displayed in Figure 6.3C. The sum of these waveforms represents again the averaged evoked potential (Figure 6.3F), however, lowpass filtered by the same overall transfer function as used in the time-varying filtering procedure (broken line in Figure 6.3A). Note that the lowpass version in Figure 6.3F is practically identical to the original waveform of Figure 6.3B, due to the high cutoff frequency (4 kHz) that has been used. Applying the computed time-varying attenuation functions (Figure 6.3D) to the bandpass outputs of Figure 6.3C results in the modified waveforms of Figure 6.3E. The addition of these waveforms produces the *time-varying filtered average* of Figure 6.3H. Those who are familiar with brainstem evoked potentials may estimate the effectiveness of the procedure by comparing this "optimally" filtered average to the normal average of Figure 6.3F.

A quantitative example of the improvements that can be achieved beyond averaging, in particular in comparison to *a posteriori* "Wiener" filtering, is presented in the companion paper. That example is based on a simulation study

*bank of bandpass filters with constant relative bandwidth (*A: *1 to 5), the average evoked potential* (B) *is filtered into subsequent partially overlapping frequency bands (*C: *1 to 5). The instantaneous (temporal) power of the* signal *and the* noise *components in these frequency bands are estimated, from which time-varying weighing functions are computed* (D, *compressed time scale). These time functions are based on the power ratio of signal and noise in the corresponding frequency bands and may vary from 0 (indicating the absence of signal power) to 1 (indicating the absence of noise power). The bandpass filtered components 1-5 of the averaged evoked potential* (C) *are attenuated in amplitude according to the corresponding weighing functions in* D *leading to the modified ("filtered") waveforms in* E. *Summation of these waveforms finally yields the* time-varying filtered ensemble average (H). *For comparison the ensemble average, lowpass filtered by the same overall transfer function as used in the t.v.f. procedure (broken line in* A*), has also been shown* (F). G *is a quasi 3D display of the weighing functions of* D *and represents the spectro-temporal weighing function* $\hat{G}(t,\dot{f}_q)$ *(see eq. (4.2)). Calibration bars apply to all waveforms except those of* D *and* G

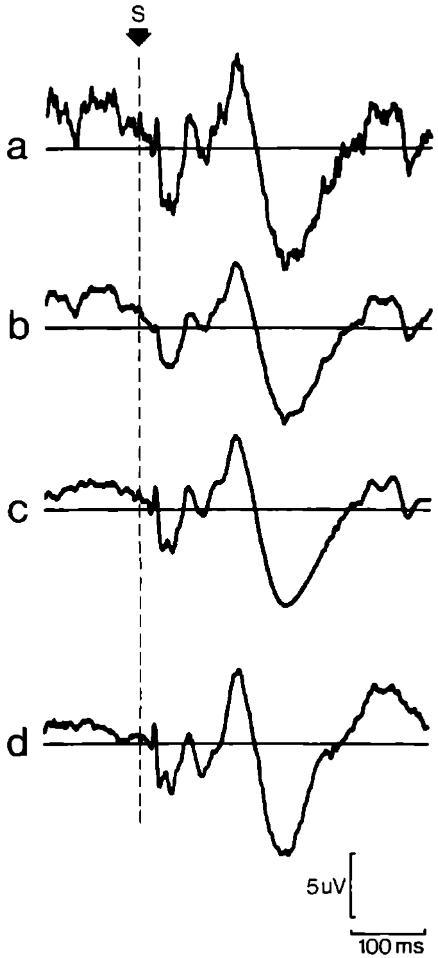


Figure 6.4a-d
Typical comparison of various analysis techniques applied to a human somatosensory evoked potential after stimulation of the median nerve at the wrist. S: moment of stimulation. a: *normal average of 20 responses;* b: a posteriori *"Wiener" filtered waveform of* a; c: a posteriori time-varying filtered *waveform of* a; d: *normal average of 10 times as many (200) responses. Note the virtual absence of "fast waves" just beyond the stimulus in* b

using an artificial signal masked by additive (stationary) noise. However, to obtain an impression of how "Wiener" and time-varying filtering compare in a *real* evoked potential situation, Figure 6.4 shows the ensemble average (20 sweeps, Figure 6.4a) of a somatosensory evoked potential; the same waveform "Wiener" (Figure 6.4b) and time-varying (Figure 6.4c) filtered, and finally again an ensemble average, but composed of 10 times as many sweeps (Figure 6.4d). Although one should be somewhat careful in interpreting the latter estimate as being closest to the "true" evoked potential (in fact, the presence of possible inhomogeneities in the ensemble due to the much longer recording session could have degraded the averaging process), the resemblance to the time-varying filtered estimate is obvious. In particular, it should be noted that the "fast" waves appearing within the first 100 ms beyond the stimulus are well preserved in the time-varying filtered estimate, while in the "Wiener" filtered estimate these are virtually absent. Using a DEC PDP 11/34 computer with floating point processor, *a posteriori* "Wiener" and time-varying filtering required additional computation times of 7 and 12 seconds respectively (figures based on a record length of 512 samples per sweep).

5. Discussion

The main objective of the present paper has been to introduce the *concept* of *a posteriori time-varying filtering*, and to place it in the general context of current evoked potential methodology. The actual elaboration of this method is presented in the companion paper (de Weerd and Kap, 1981b). For a clear understanding of this new technique, it was necessary to describe the assumed underlying evoked potential model in detail. According to that model, an evoked response ensemble is considered as a mixed process consisting of a deterministic transient signal and additive (nonstationary) noise. It was further pointed out that in the ensemble a clear distinction should be made between local and global nonstationarity (cf. Figure 6.1). To avoid terminological confusion it was proposed to use the notions *nonstationarity versus inhomogeneity* to distinguish between these two types.

Assuming then a homogeneous nonstationary ensemble, the question of optimal signal estimation (in the sense of least mean-square error) was investigated. Inspection of the pertinent literature on communication engineering revealed that, as far as we were able to determine, no readily applicable theory is at present available. On the other hand, by considering the estimation problem from the statistical point of view, significant additional insight into the problem could be obtained. Firstly, and perhaps most importantly, it was established that estimation of a sampled evoked potential waveform is, in fact, a simultaneous (multidimensional) estimation problem, rather than a number of independent point estimation problems (each point corresponding with the individual samples of that waveform). The importance of this conclusion stems from the explicit proof, (Stein, 1955), that for simultaneous estimation problems better estimators than ensemble averaging (which is essentially an individual point estimation technique) exist. Thus the controversial question of whether or not it is possible to improve beyond averaging has apparently been solved, at least from the theoretical point of view.

The papers of Strackee (1979) and Nagelkerke and Strackee (1979), and not least of all the disputes with these authors (Nagelkerke, 1979) further led to the observation, that the method of *"a posteriori* 'Wiener' filtering" *(a.p.w.f.)* shows a striking resemblance to elaborated statistical procedures known as *empirical Bayes estimators* (Efron and Morris, 1972; 1973). Although the precise relation between a.p.w.f. and these estimators requires further theoretical investigation, it could be made *plausible*, using theoretical arguments, that a.p.w.f. belongs to the class of estimators which may give an improved estimation beyond averaging, a fact that hitherto had only been suggested by simulation studies (de Weerd and Martens, 1978; Kearney, 1979). In particular, it had been doubted whether or not *second order functionals* estimated from the data (namely the estimated spectra of signal and noise) could ever be usefully implemented to improve the estimated *first order functional* (the ensemble average) of the very same data. The theory of empirical Bayes estimators clearly indicates that such estimated knowledge can be used in a profitable way. Of course, less optimal estimation schemes result than those based on *a priori available* knowledge. However, these sub-optimal schemes may still remain significantly better than ensemble averaging.

The foregoing discussion should not be understood as a renewed plea for the application of a.p.w.f. to averaged evoked potentials. Rather, it aims at explaining the motivation for developing a generalized "Wiener" filtering, which would, unlike a.p.w.f., be suitable to handle the typically transient structure of evoked potential waveforms. This could be accomplished by generalizing the "Wiener" filter transfer function to a weighing function in the combined time-frequency domain. This weighing function in turn is based on time-varying spectra, which are adapted to the evoked potential structure in time and frequency, such that for "slow" waves the relative emphasis is on precise frequency content, rather than on time resolution while for "fast" waves the emphasis is on precise location in time, rather than on spectral resolution. Although the resulting *time-varying filtering (t.v.f.)* procedure is therefore based on global characteristics of evoked potential waveforms, it should be emphasized that no *a priori* knowledge of the actual waveshape or latencies of waveform components is assumed.

The principle of estimating *a posteriori* information from the ensemble, as employed in the t.v.f. method, is similar to that of a.p.w.f. It is interesting to note that in practice where we deal with sampled signals, the time-varying weighing function even contains precisely as many independent coefficients as the transfer function of the "Wiener" filter would hold. The crucial difference, however, is that "Wiener" filtering considers the signal-to-noise power ratio entirely in the frequency domain (thus implicitly assuming time-*in*variant waveforms), whereas time-varying filtering takes the *temporal* distribution of this ratio into account as well, thereby sacrificing needless spectral resolution, especially in the higher frequency region. Accordingly t.v.f. is able to differentiate between signal and noise components which share the same frequency band but differ in temporal occurrence. Herein lies the main strength of the newly proposed method, and it explains why it is a usual finding that substantial improvement over averaging can be obtained where a.p.w.f. fails to do so. Ultimately this is the consequence of the fact that the transient, dynamic waveform structure of evoked potentials calls for a dynamic, time-varying, filtering, rather than a time-*in*variant filtering approach; a conclusion which is supported by several years of experience in applying the actual t.v.f. method to peripheral and central human evoked potentials.

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Chapter 7

A Posteriori Time-varying Filtering of Averaged Evoked Potentials

II. Mathematical and Computational Aspects

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Abstract

The problem of estimating an unknown transient signal, given an ensemble of waveforms, in which this signal appears as a nonrandom component in the presence of additive noise is considered. This problem is solved by generalizing the method of "*a posteriori* 'Wiener' filtering". In the new method, the ensemble average is filtered by a *time-varying* system which is based on estimated time-varying power spectra of signal and noise. The nature of this system, and the computational procedures involved, are discussed in detail. A software package for time-varying filtering is briefly described. Application of the method is illustrated by a simulation example, which also provides a comparison to time-*in*variant *a posteriori* "Wiener" filtering.

1. Introduction

This paper deals with the estimation of an unknown transient signal on the basis of an ensemble of waveforms in which the signal is assumed to be an identically repeating, nonrandom component, masked by additive noise. Although the formulation of the problem has mainly been inspired by electrophysiological experiments, notably by the measurement of *evoked potentials*, the same question arises in many other applications of practical interest.

Evoked potentials from the brain can be obtained by applying a sensory stimulus, such as a flashing pattern, an acoustic click or an electric shock, to a subject and recording the resultant electrical activity from the scalp using surface electrodes. On application of an electric shock to e.g. the fingers or toes, peripheral and spinal evoked potentials may also be recorded (see Desmedt, 1980, for a review). These potentials may be considered as impulse responses of their corresponding electrophysiological system and their waveforms are, typically, of a transient character (see de Weerd and Kap, 1981). By applying the stimulus repeatedly, an *ensemble* of responses $\{\underline{x}_i(t)\}$ is obtained which may be conceptually modelled as

$$x_i(t) = s(t) + n_i(t) \qquad i = 1,2, \ldots\ldots N; \qquad 0 \leq t \leq T \tag{1.1}$$

where s(t) represents the desired evoked potential (the "signal"), $n_i(t)$ the i-th ensemble element of the disturbing process (the "noise", mainly determined by the background EEG, other bioelectric activity, mains interference and instrumental noise), N the number of stimulus repetitions and T the duration of the evoked potential. In terms of communication theory (1.1) is a *mixed process*, since it is essentially stochastic, but contains in addition a deterministic component. Even if the noise were assumed to be stationary with zero mean, the mean value of the process (i.e. the signal component) varies with (lag-)time t and therefore the process is *nonstationary*. But we will assume that the ensemble is *homogeneous*, which implies that the statistical properties do not vary *across the ensemble*. (For a discussion on the fundamental difference between statistical variations across lag time (nonstationarity) and across the ensemble (inhomogeneity) the reader is referred to the companion paper (de Weerd, 1981).) In practice, the assumed homogeneity is of course only a rough approximation, since both s(t) and the statistical properties of n(t) vary over long intervals of (real) time, due to the inherent variability of the generating system. However, we will assume that the total duration of the experiment wherein the ensemble $x_i(t)$ is obtained is short enough, so that changes in s(t) and the statistical properties of n(t) can be neglected. Typical values for N may range from a few thousands for the Brainstem Auditory Evoked Potential *(BAEP)* to a few hundreds for the Somatosensory Evoked Potential *(SSEP)* or even a few tens for the Visual Evoked Potential *(VEP)*. The evoked potential duration T varies from approximately 10 ms (BAEP) to about 500 ms (SSEP and VEP).

The usual practice of estimating s(t) is by making up the ensemble average

$$\bar{x}(t) = s(t) + \frac{1}{N} \sum_{i=1}^{N} n_i(t) \tag{1.2}$$

It can be proved that this estimator does *not* provide the best estimate in the mean-square error sense. Better estimators exist which do not require any additional *a priori* information on the signal and the statistical structure of the noise. In the companion paper (de Weerd, 1981) these problems were dealt with in some detail. It was concluded that, in the case of non-transient signals, the method of *a posteriori* "Wiener" filtering (Walter, 1969; Doyle, 1975) does indeed lead to a further improvement beyond averaging. Moreover it was postulated that this method can be generalized to *time-varying filtering (t.v.f.)*, for application to transient signals. In the latter case, the ensemble average $\bar{x}(t)$ is passed through a time-varying filter, which is calculated from estimated time-varying power spectra of signal and noise. In the following discussion, this idea is worked out in detail. The estimation of signal and noise spectra from the ensemble is dealt with in section 2. Section 3 starts with some considerations concerning time-varying power spectra, after which the actual t.v.f. procedure is elaborated. Section 4 describes a t.v.f. software package which is also used in the simulation example of section 5. In that example, results of time-varying filtering are compared to time-*in*variant *a posteriori* "Wiener" filtering and to "theoretical" filtering, based on the *a priori* available knowledge of the signal and the statistical structure of the noise. Finally, section 6 provides some concluding theoretical and practical comments.

2. Generalization of *a posteriori* "Wiener" Filtering

The estimated transfer function of the *a posteriori* "Wiener" filter for use on the average evoked response $\bar{x}(t)$ is given by (Walter, 1969; Doyle, 1975; de Weerd and Martens, 1978):

$$\hat{H}(f) = \frac{\Phi_{ss}(f)}{\Phi_{ss}(f) + \frac{1}{N}\Phi_{nn}(f)} \tag{2.1}$$

where $\Phi_{ss}(f)$ and $\Phi_{nn}(f)$ are the *estimated* power density spectra of signal and noise and N the number of averaged ensemble elements. Walter (1969) described a method for estimating the spectra $\Phi_{ss}(f)$ and $\Phi_{nn}(f)$ from the ensemble (1.1). We will follow, however, another approach, which allows more efficient computational procedures (de Weerd et al., 1979). That method is, in principle, based on the composition of two power density spectra:

- the spectrum of the ensemble average $\bar{x}(t)$, denoted by $\Phi_{\bar{x}\bar{x}}(f)$. The expected value of that spectrum consists of the signal power density spectrum plus the noise power density spectrum, the latter reduced by a factor proportional to the number of averaged ensemble elements N;
- the spectrum of the *alternate* averaged ensemble $\tilde{x}(t)$, denoted by $\Phi_{\tilde{x}\tilde{x}}(f)$. Due to the technique of averaging with alternate polarity, that is, the alternate addition and subtraction of ensemble elements, the signal cancels out and the power density spectrum is an estimate of the pure noise spectrum, again reduced by a factor N.

It has been shown that an improved estimate of the latter spectrum can be ob-

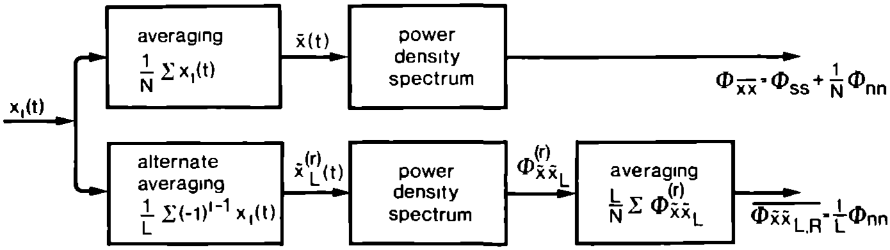

Figure 7.1
Procedure for the estimation of the power spectra of signal and noise, given an ensemble of waveforms in which the signal appears as a deterministic component in the presence of additive noise. In the upper channel the ensemble (of size N) is averaged and the corresponding power spectrum calculated. In the lower channel the same ensemble is divided into subensembles (r=1,2, ...R) of size L. In each subensemble the alternate average and its corresponding power spectrum are computed. The resulting R (=N/L) spectra are averaged in order to reduce the variance. From the spectra thus obtained the estimated underlying power spectra of the signal (Φ_{ss}) and the noise (Φ_{nn}) can be calculated by simple mathematics

tained by splitting up the ensemble into several *subensembles* (Figure 7.1). In this case we obtain a spectrum $\overline{\Phi_{\tilde{x}\tilde{x}_{L,R}}(f)}$ where the subscript L stands for alternate ensemble averaging over L responses (the size of the subensembles) while the subscript R indicates averaging over the power density spectra, computed from these subensembles, i.e.

$$\overline{\Phi_{\tilde{x}\tilde{x}_{L,R}}(f)} = \frac{1}{R}\sum_{r=1}^{R} \Phi_{\tilde{x}\tilde{x}_L}^{(r)}(f) \tag{2.2}$$

where the total number of subensembles R follows from R = N/L. For further details of this procedure, the reader is referred to de Weerd et al., 1979 (their Figure 2).From the power density spectra $\Phi_{\overline{xx}}(f)$ and $\overline{\Phi_{\tilde{x}\tilde{x}_{L,R}}(f)}$ we may thus obtain estimates of the power density spectra of the signal and the noise (see also Figure 7.1):

$$\Phi_{ss}(f) = \Phi_{\overline{xx}}(f) - \frac{1}{R}\,\overline{\Phi_{\tilde{x}\tilde{x}_{L,R}}(f)} \tag{2.3}$$

and

$$\Phi_{nn}(f) = L\,\overline{\Phi_{\tilde{x}\tilde{x}}(f)}_{L,R} \tag{2.4}$$

Substituting this into (2.1) and recalling that N=R·L we obtain

$$\hat{H}(f) = 1 - \frac{1}{R}\frac{\overline{\Phi_{\tilde{x}\tilde{x}}(f)}_{L,R}}{\Phi_{\overline{xx}}(f)} \tag{2.5}$$

as the estimated transfer function of the (time-invariant) *a posteriori* "Wiener" filter.

For a nonstationary, but homogeneous, ensemble,(2.5) may be generalized to a time-varying weighing function $\hat{G}(t,f)$ by conceptually replacing the power density spectra by their corresponding time-varying counterparts. We thus obtain

$$\hat{G}(t,f) = 1 - \frac{1}{R}\frac{\overline{\Phi_{\tilde{x}\tilde{x}}(t,f)}_{L,R}}{\Phi_{\overline{xx}}(t,f)} \tag{2.6}$$

where the symbol t refers to lag-time beyond the stimulus.

It should be noted that both $\hat{H}(f)$ and $\hat{G}(t,f)$ are, in their present form, inappropriate for direct application. In particular, the spectra appearing in (2.5) and (2.6) require proper smoothing, and possibly occurring negative values in $\hat{H}(f)$ and $\hat{G}(t,f)$ should be clipped to zero (cf. de Weerd and Martens, 1978). These problems, and the further elaboration of (2.6) will be dealt with in the next section, once a proper means for obtaining time-varying spectra has been established.

3. Time-varying Filtering

In an earlier paper (de Weerd and Kap, 1981) we have dealt with the question of obtaining *time-varying power spectra* of evoked potentials. In that paper it is concluded that a suitable method for obtaining these spectra is by using a bank of bandpass filters of constant relative bandwidth. Such a method is well matched to the typical time-frequency structure of evoked potentials, consisting of "early" components of relatively short duration, high frequency and large bandwidth, and "later" components of longer duration, low frequency and small bandwidth. We will further assume that the global time-frequency structure of the background noise is not vastly different from that of the evoked potentials, so that the above method for obtaining time-varying spectra is applicable to the noise as well.

When using a bank of bandpass filters, the desired time-varying power

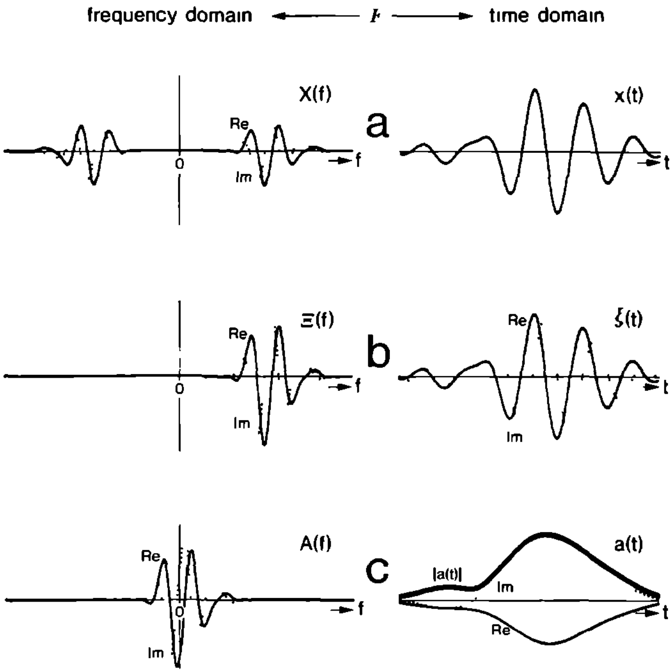

Figure 7.2a-c *Representation of bandpass signals and computation of the temporal envelope (solid curves represent* real, *dotted curves* imaginary *parts).* a: *the Fourier transform $X(f)$ of a real signal $x(t)$ contains both positive and negative frequencies, related as $X(-f) = X^*(f)$.* b: *the one-sided spectrum $\Xi(f)$ is obtained by omitting the negative frequencies and doubling the magnitude of the positive ones. Its inverse Fourier transform represents the* analytic signal *$\xi(t)$, which can also be obtained by adding to the original real signal an imaginary component which is the Hilbert transform of $x(t)$.* c: *translation of the one-sided spectrum to zero frequency corresponds to complex demodulation in the time domain. This leads to the* complex envelope *$a(t)$, whose modulus represents the* temporal envelope (heavy line) *of the original real signal $x(t)$*

spectrum can be obtained by computing the squared magnitudes of the *complex envelopes* of the individual filter outputs. Although the rationale of this method has been explained in the above cited paper, we will recapitulate its basic principles, since these will be needed in the elaboration of the time-varying filtering technique as well. To this end, we will first consider the description of bandpass filtered signals and their temporal (instantaneous) power, using the complex signal representation.

3.1 Representation of Bandpass Signals [1)]

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1) For references to the pertinent literature on this subject see de Weerd and Kap, 1981 (their Appendix).

Let a real bandpass signal be given by $x(t)$ and its Fourier transform by $X(f)$ (Figure 7.2a). Since $x(t)$ is real, we have $X(-f) = X^*(f)$, where the starring denotes the complex conjugate. We define the one sided spectrum $\Xi(f)$ such that

$$\begin{aligned}\Xi(f) &= 2X(f) \qquad f > 0\\ &= 0 \qquad\qquad f < 0\end{aligned} \tag{3.1}$$

(Figure 7.2b), so that

$$X(f) = \tfrac{1}{2}\{\Xi(f) + \Xi^*(-f)\} \tag{3.2}$$

Denoting the mean frequency of the passband of $\Xi(f)$ by f_o, we further define $A(f)$, a frequency translated version of $\Xi(f)$, by

$$A(f) = \Xi(f+f_o) \tag{3.3}$$

(Figure 7.2c). Inserting (3.3) into (3.2) it follows that

$$X(f) = \tfrac{1}{2}\{A(f-f_o) + A^*(-f-f_o)\} \tag{3.4}$$

The equivalent relations in the time domain follow by using some well-known properties of the Fourier transform [1]. Thus, from (3.2) we have:

$$x(t) = \tfrac{1}{2}\{\xi(t) + \xi^*(t)\} = \mathrm{Re}\{\xi(t)\} \tag{3.5}$$

where $\xi(t)$ is the inverse Fourier transform of $\Xi(f)$ and Re{ } means "real part of". Likewise, we obtain from (3.4):

$$x(t) = \tfrac{1}{2}\{a(t)\exp(2\pi i f_o t) + a^*(t)\exp(-2\pi i f_o t)\} = \mathrm{Re}\{a(t)\exp(2\pi i f_o t)\} \tag{3.6}$$

where $a(t)$ is the inverse Fourier transform of $A(f)$. We note that $\xi(t)$ is generally referred to as the *analytic signal* of $x(t)$. It is a complex function of time, its imaginary part being the *Hilbert transform* of $x(t)$. The time function $a(t)$ is usually termed the *complex envelope* of $x(t)$. By splitting $a(t)$ into its modulus and phase

$$a(t) = |a(t)| \exp(i\phi(t)) \tag{3.7}$$

and substituting this into (3.6) we find:

$$x(t) = \mathrm{Re}\{|a(t)|\exp(2\pi i f_o t + i\phi(t))\} = |a(t)|\cos(2\pi f_o t + \phi(t)) \tag{3.8}$$

where it is seen that $x(t)$ can be regarded as a frequency and amplitude modulated signal with *envelope* $|a(t)|$. The squared value $|a(t)|^2$ thus represents

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[1] If the inverse Fourier transform of $Z(f)$ is $z(t)$, then the transform of $Z^*(-f)$ is $z^*(t)$ and the transform of $Z(f-f_o)$ equals $z(t)\exp(2\pi i f_o t)$.

the *instantaneous (temporal) power* of x(t).

The *complex envelope* a(t) not only provides a useful means for determining the instantaneous power of a bandpass signal, but it also allows a more efficient *sampling* of that signal. This can be illustrated as follows. Assuming a real bandpass signal with a bandwidth B, centered around a mean frequency f_o, an adequate sampling rate should be at least $2f_o + B$ samples per second. However, (3.6) shows that a bandpass signal is uniquely determined by its complex envelope a(t) and the "carrier frequency" f_o. The complex envelope is a lowpass signal (cf. Figure 7.2c), which can be sampled at a rate of only B (complex) samples per second.

Actually the envelope $|a(t)|$ can be determined directly by sampling the analytic signal $\xi(t)$ (cf. Figure 7.2b) at the reduced rate of B (complex) samples per second, so that the step of complex demodulation (leading to the complex envelope a(t)) is, in fact, unnecessary. However, in digital signal processing the analytic signal is usually obtained via the frequency domain, so that a reduction of the sampling rate is much easier achieved by means of a translation and decimation of samples in the frequency domain, than by discarding intermediate samples in the time domain. In practice, the former procedure includes the following steps: *(1)* Fourier transformation of the signal; *(2)* omitting the negative frequencies and doubling the magnitude of the positive ones (cf. (3.1)); *(3)* translating this spectrum to zero frequency (cf. (3.3)) and *(4)* applying the inverse Fourier transform with a reduced number of samples. As stated before, the squared magnitude of the complex envelope thus obtained represents the instantaneous power of the original bandpass signal.

3.2 Actual Filtering Procedure

When using a *bank of bandpass filters* for the estimation of a time-varying power spectrum, we obtain a set of time-varying *squared envelopes*, each corresponding to a particular section in frequency (see de Weerd and Kap, 1981; their Figure 4). The output of such a time-varying spectral analysis system can be written in the general form

$$\Phi(t,f_q) = |a_q(t)|^2 \qquad q = 1,2, \ldots\ldots Q \tag{3.9}$$

where f_q represents the center frequency of the q-th bandpass filter in a bank of Q filters and $a_q(t)$ is the complex envelope of the output of the same filter.

The two time-varying power spectra which are needed for the calculation of the time-varying weighing function $\hat{G}(t,f)$ (cf. (2.6)) are obtained by passing the ensemble average $\bar{x}(t)$, and the alternate subensemble averages $\tilde{x}_L(t)$ through the filter bank. Denoting the complex envelopes of $\bar{x}(t)$ and $\tilde{x}_L(t)$ in the q-th passband by $\bar{a}_q(t)$ and $\tilde{a}_q(t)$ respectively, and inserting (3.9) into (2.6) we obtain:

$$\hat{G}(t,f_q) = \hat{g}_q(t) \qquad q = 1,2, \ldots\ldots Q \tag{3.10}$$

with

$$\hat{g}_q(t) = 1 - \frac{1}{R}\,\frac{\overline{|\tilde{a}_q(t)|^2}}{|\bar{a}_q(t)|^2} \tag{3.11}$$

where $\hat{g}_q(t)$ represents the estimated time-varying weighing function in the q-th passband and the bar across the numerator represents, as in (2.6), averaging over R subensembles.

The squared envelopes appearing in (3.11) are *estimated* envelopes with inherent variabilities. This especially applies to the squared envelopes of the ensemble average which appear in the denominator [1]. In this respect the situation is completely analogous to the estimated *a posteriori* "Wiener" filter transfer function (cf. de Weerd and Martens, 1978). As a consequence, $\hat{G}(t,f_q)$ will, in general, show a large variance, and negative values will be likely to occur as well, in particular at low signal-to-noise ratios. It is therefore essential that the raw squared time envelopes $\overline{|\tilde{a}_q(t)|^2}$ and $|\bar{a}_q(t)|^2$ are properly *smoothed*, prior to the calculation of the weighing functions $\hat{g}_q(t)$. This may be done either directly, by convolution with a smoothing window in the time domain, or indirectly, by multiplying the spectrum of the squared time envelopes by a lowpass filter function in the frequency domain. These techniques are comparable to the well-known techniques of smoothing power density spectra (see e.g. Jenkins and Watts, 1968).

The *degree* of smoothing need not be the same in all Q passbands since higher frequency bands generally require heavier smoothing due to the lower "signal-to-noise" ratios in these bands. However, heavier smoothing leads to a loss of effective temporal resolution in the squared envelopes and thus injures the time-varying aspect of the filtering method, so that obviously some compromise is necessary. No general rules for determining optimal values for

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[1]) The variance of the squared envelopes of the alternate averages can be made small by averaging over sufficient subensembles.

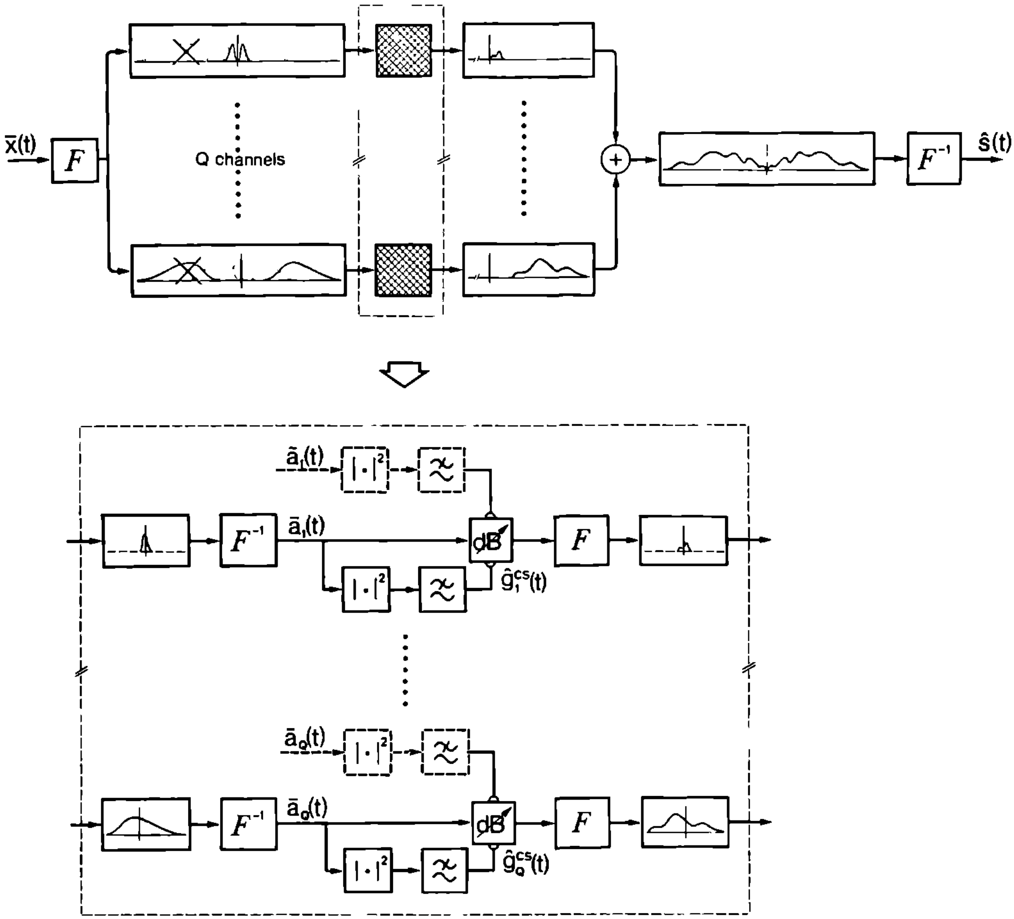

Figure 7.3

Block diagram of the actual time-varying filtering *procedure. The application of this procedure supposes that two types of averages have been made, namely a* normal *ensemble average $\bar{x}(t)$ and an* alternate *ensemble average $\tilde{x}(t)$. (In practice, several alternate* subensemble *averages are obtained, cf. section 2, but this aspect is omitted for reasons of clarity.) The ensemble average $\bar{x}(t)$ is filtered into subsequent frequency bands (q= 1,2, ... Q) of proportionally increasing width. In each of the bands complex envelopes $\bar{a}_q(t)$ are obtained by omitting the negative frequencies, translating the spectra to zero frequency and inverse Fourier transformation. In exactly the same manner (*not shown*), by passing the alternate ensemble average $\tilde{x}(t)$ through the same bank of bandpass fil-*

the smoothing parameters exist; this remains mainly a matter of trial and error. However, these parameters are not very critical with respect to the final result and it suffices to establish their values upon entering a new field of applications, rather than adapting them to each individual problem at hand. In fact, three sets of parameter values have proved to cover almost all applications in practice (see section 4).

In spite of adequate smoothing, the resulting weighing functions, denoted by $\hat{g}^s_q(t)$, may still exhibit negative values at extremely low signal-to-noise ratios, again similar to the a.p.w.f. transfer function. In such cases erroneous results are avoided by "clipping" these values to zero. We thus have

$$\hat{g}^{cs}_q(t) = \max\{0, \hat{g}^s_q(t)\} \tag{3.12}$$

and, as our final time-varying weighing function:

$$\hat{G}^{cs}(t,f_q) = \hat{g}^{cs}_q(t) \qquad q = 1,2, \ldots\ldots Q \tag{3.13}$$

where the superscripts *s* and *c* stand for "smoothed" and "clipped" respectively. From the foregoing considerations it follows that the proposed system $\hat{G}^{cs}(t,f_q)$ can be physically characterized as a bank of fixed bandpass filters followed by time-varying attenuators. The desired output of this system, that is, the time-varying filtered ensemble average may be simply obtained by passing the input $\bar{x}(t)$ through the filter bank, applying the appropriate $\hat{g}^{cs}_q(t)$ to each of the bandpass filters' outputs and summing all contributions (see the companion paper: Figure 2 and eq. (4.4)):

ters, complex envelopes $\tilde{a}_q(t)$ of the alternate average are obtained. The squared magnitudes of the complex envelopes $\bar{a}_q(t)$ and $\tilde{a}_q(t)$ are used, after appropriate smoothing, to calculate time-varying weighing functions *(cf. (3.11)), which are essentially based on the estimated temporal signal-to-noise power ratio in the corresponding frequency bands. After possibly negative values have been clipped to zero (not shown), these weighing functions are applied to the complex envelopes $\bar{a}_q(t)$ of the normal average. The final time-varying filtered estimate is reconstructed by following part of the above procedure in an inverse way, namely by Fourier transformation of the* modified *complex envelopes, translation of their spectra to the original bandpass filter center frequencies, recombination of all frequency bands, addition of negative frequencies in a conjugate symmetrical manner* (shown for real part only) *and inverse Fourier transformation*

$$\hat{s}(t) = \sum_{q=1}^{Q} \hat{g}^{cs}_{q}(t)\ \bar{x}_q(t) \tag{3.14}$$

where $\hat{s}(t)$ is the time-varying filtered ensemble average and $\bar{x}_q(t)$ is the output of the q-th bandpass filter when $\bar{x}(t)$ is applied as input.

In view of the discussion in section 3.1, however, it is computationally more efficient to replace the bandpass filtered signals $\bar{x}_q(t)$ by their complex envelopes $\bar{a}_q(t)$. Thus we obtain:

$$\hat{s}(t) = \sum_{q=1}^{Q} \hat{g}^{cs}_{q}(t)\ \mathrm{Re}\{\bar{a}_q(t)\ \exp(2\pi i f_q t)\} \tag{3.15}$$

where f_q represents, as before, the center frequency of the q-th bandpass filter. Since $\hat{g}^{cs}_{q}(t)$ is a purely real function (3.15) can be rewritten as

$$\hat{s}(t) = \mathrm{Re}\{\sum_{q=1}^{Q} \hat{g}^{cs}_{q}(t)\ \bar{a}_q(t)\ \exp(2\pi i f_q t)\} \tag{3.16}$$

Equation (3.16) provides, with (3.11) and the subsequent smoothing and clipping modifications of the functions $\hat{g}_q(t)$, the principles of the time-varying filtering method. A block diagram of the actual procedure is shown in Figure 7.3. The ensemble average $\bar{x}(t)$ is bandpass filtered into subsequent frequency bands ("channels") and in each frequency band complex envelopes $\bar{a}_q(t)$ are obtained in the manner discussed in section 3.1. Each complex envelope is multiplied by the appropriate weighing function $\hat{g}^{cs}_{q}(t)$. The modulation operation of (3.16) is carried out in the frequency domain by translating the spectra of the modified complex envelopes by an amount f_q, whereafter all frequency bands are added. Likewise, the "real part" operation of (3.16) is performed in the frequency domain, by adding the negative frequencies in a conjugate symmetric manner (cf. (3.2) and (3.5); time domain operations have been replaced by equivalent operations in the frequency domain for reasons of ease and speed of computation). The final time-varying filtered estimate is then obtained after inverse Fourier transformation.

3.3 Further Relevant Aspects

In the foregoing subsection, the characteristics of the bank of bandpass filters used for the estimation of the time-varying power spectra, have not been specified in detail, and we will comment on it in what follows. In a previous paper (de Weerd and Kap, 1981) it has been pointed out that an adequate method for obtaining time-varying power spectra requires the use of bandpass filters with good spectral and temporal properties. It was shown that filters having a transfer function of a (nonsymmetrical) cosine shape perform well in this re-

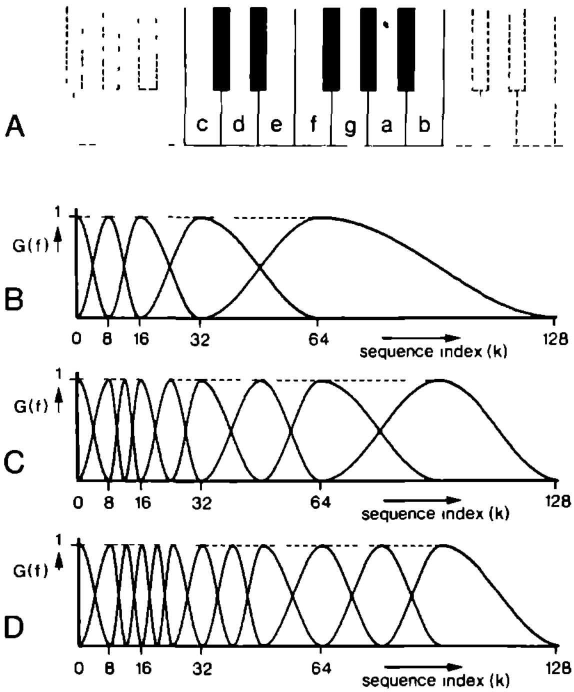

Figure 7.4A-D
When a bank of bandpass filters with (approximately) constant relative bandwidth is to be implemented on a digital computer, it is convenient if the central frequencies of subsequent filters are related in some "nice" manner. With the piano keyboard (A) *in mind it is easily recognized that* tonal *and thus nicely interrelated chords are: the basic octave (keys c-c-etc.); fifth octave (keys c-g-c-etc.); and third fifth octave (keys c-e-g-c-etc.). These correspond with frequency ratios 1-2-4-etc.* (B); $1-\frac{3}{2}-2$*-etc.* (C); *and* $1-\frac{5}{4}-\frac{3}{2}-2$*-etc.* (D). *Sequence indices, rather than absolute frequencies have been indicated, since the latter depend, of course, on the sampling frequency actually used. Note that in the lowest frequency region the constant relative bandwidth property does not hold. This is due to the fact that a minimum number of samples is required for an adequate description of individual filter transfer functions (cf. de Weerd and Kap, 1981)*

spect. However, the fundamental problem remains that spectral and temporal resolution are inversely related quantities, so that the actual choice of a filter bank is a matter of compromizing between the desired resolution in time and frequency.

In the class of *proportional bandwidth* filters, a first practical choice is a bank of octave bandpass filters, but by analogy with musical theory we have tried several other banks (Figure 7.4). These were used both in simulation studies and in the estimation of real evoked potentials. Of course the banks containing more filters with narrower bandwidth provide a higher frequency resolution, at the cost of a lower time resolution. However, application of these banks led to only minor differences in the final time-varying filtered results which were compared on the basis of their respective mean square errors (simulations) or by visual inspection of the different components of the evoked potential waveforms. It is thus concluded that a bank of octave bandpass filters is a simple, and yet appropriate choice.

Once this choice is made, it should be noted that the overall transfer function of the filter bank does not pass the entire frequency band, since the highest filter in the bank gradually falls to zero, starting halfway the entire passband (cf. Figure 7.4B). In practice this is no problem since evoked potentials are usually largely oversampled, in order to enhance the visibility of the smaller waveform components and increase the estimation accuracy of their time of occurrence. Occasionally it may even be advisable to leave out one or more of the higher filters in the bank, thus limiting the bandpass of the system and further improving the signal-to-noise ratio. Since digital, zero-phase, filters are used such a shrinkage of the system bandpass can be done without introducing phase distortion into the signal. On the other hand, if one expects significant signal contributions in the high frequency range, some oversampling of the data prior to application of time-varying filtering is advisable. The fact that higher frequencies are not entirely passed should also be taken into consideration when time-varying filtering is compared to other methods of analysis (e.g. normal ensemble averaging). In this case objective comparison is only possible when the data, subjected to other methods, is prefiltered by a bandpass filter having the same overall transfer characteristics as the one used in the time-varying filtering procedure.

Another point of concern is the fact that time-varying filtering makes extensive use of the Discrete (Fast) Fourier Transform (*DFT)*, in order to perform the various filtering operations that are involved. Consequently, one is faced with some inherent DFT problems, in particular with the occurrence of periodic, or circular, convolution (see e.g. Bergland, 1969). Due to the periodic nature of the discrete Fourier transform, it appears as if the beginning and the end of a sampled time signal are tied together, so that they interact when the signal is filtered via the frequency domain (the so-called "wrap-around" effect). One possible, and relatively simple, way to overcome that problem is by augmenting the total observation interval (say, by 25%) prior to and beyond the actual interval of interest, and to discard the augmented parts upon completion of the time-varying filtering procedure. In evaluating the technique of *a posteriori* "Wiener" filtering, Ungan and Basar (1976) noted that extension of the observation interval beyond the expected duration of the signal seriously hampers the effectiveness of that procedure (see also de Weerd and Kap, 1981). It should be stressed that such limitations, which are inherent to time-*in*variant methods, do not apply to the method of time-varying filtering, for the obvious reason that the *temporal* distribution of the signal power is taken into account as well. Thus, in principle, arbitrarily large pre- and post-stimulus intervals may be chosen. This appears to be an important advantage in evoked potential estimation where the duration of the various waveform components differ widely and are usually unknown beforehand (cf. Naitoh and Sunderman, 1978).

4. A Software Package for Time-varying Filtering

Based on the block diagram in Figure 7.3, a software package for time-varying filtering (t.v.f.) has been developed. This package, written in standard Fortran IV and originally implemented on a DEC PDP 11/34 basically consists of three major subroutines (which in turn call for five smaller auxillary subroutines) [1]. The three major routines are: *(1)* a t.v.f. initialization routine, *(2)* a "sweep" routine and *(3)* a final time-varying filtering routine. Two versions of the initialization routine are available. In the first version all that needs to be specified is the number of samples per sweep (K), the number of sweeps per subensemble (L) and the desired smoothing of the squared envelopes (minor, nominal, or heavy). The program then constructs the bank of bandpass filters, the number of filters being dependent on the number of samples per sweep (see de Weerd and Kap, 1981), and it selects the proper smoothing parameters per filter. This initialization routine is used in standard applications. In the second version the user may specify his own bank of bandpass filters of arbitrary widths (although the transfer functions remain of the cosine type); he may also leave out any filters at will. Furthermore for each filter in the bank the user must specify a separate smoothing parameter. This routine is designed for special applications, such as the selective filtering of specific frequency components in the input data. The "sweep" routine, which has no input parameters, processes subsequently acquired sweeps. It records the number of incoming sweeps and undertakes the necessary computational steps upon completion of each subsequent subensemble. The final t.v.f. routine, which also has no input parameters, produces the time-varying filtered average, along with the normal ensemble average.

From the foregoing description it should be clear that the computer implementation of this t.v.f. package is simple and straightforward, so that it can be interfaced with virtually any current averaging program. At present the

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[1] This modular package is available from the authors.

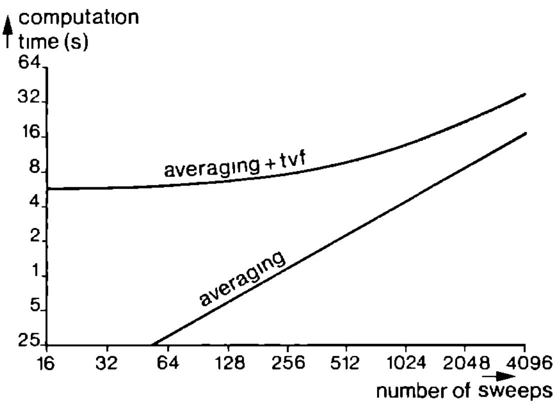

Figure 7.5
Total computation times for time-varying filtering (including averaging) compared to averaging alone, as a function of the number of sweeps, using a constant number of subensembles (8) and 256 samples per sweep. Computation times depend almost linearly on the latter number. (Figures based on a PDP 11/34 with floating point hardware)

program size is, apart from FFT routines, approximately 2200 (integer) words, while the required additional data storage (integer words) amounts to 7 times the number of samples per sweep. Figure 7.5 shows typical computation times [1)] for the overall t.v.f. procedure (including averaging) *vs.* the number of sweeps, using 8 subensembles and 256 samples per sweep. (Computation times depend almost linearly on the latter number.) For comparison the required time for normal averaging has also been shown.

5. A Simulation Example

When applying time-varying filtering it is impossible to make general statements about the improvement that can be obtained beyond averaging. This improvement depends on such complicated factors as the spectro-temporal structure of signal and noise, and their relative intensities. However, to obtain some idea regarding the capabilities of the method we will describe a simulation example with some practical relevance.

5.1 Computational Procedure

In this example we have chosen an artificial signal which shows resemblance to a somatosensory evoked potential (Figure 7.6, dotted curves). The noise was obtained from a hardware noise generator (Hewlett Packard model 3722A) and filtered such that its bandwidth resembled that of the normally encountered background. This was judged both by visual inspection of averaged signal-plus-noise waveforms and by spectral analysis. By using filtered noise more realis-

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1) Based on a DEC PDP 11/34 computer with hardware floating point processor.

tic results are obtained than if "white" noise had been used. Sometimes in evoked potential experiments a short term background (noise) suppression is seen upon presentation of a stimulus. For simplicity this effect has not been included and stationary noise was used. (Note, however, that such an effect would favour the t.v.f. method as compared to time-*in*variant filter methods, such as "Wiener" filtering.)

Ensembles of 128 "sweeps" were constructed by adding the signal to subsequent realizations of the noise. Three different initial signal-to-noise power ratios (defined as the ratio of the integrated square value of the signal and the ensemble mean of the integrated square value of the noise over the time-interval shown in Figure 7.6) were used, namely -25, -15, and -5 dB. These ensembles were averaged and filtered with a bandpass filter having the same overall transfer function as the one used in the time-varying filtering procedure (cf. section 3.3). Results are shown in Figure 7.6a. Next, time-varying filtering, as described in section 3.2, was applied (Figure 7.6d). In order to provide a more general basis for comparison, another three filters were also applied to the same ensemble average, namely smoothed *a posteriori* "Wiener" filtering (Figure 7.6b) and "theoretical" Wiener filtering (Figure 7.6c, see de Weerd and Martens, 1978), and "theoretical" time-varying filtering (Figure 7.6e). The latter two filters can be computed by virtue of the fact that, in a simulation example, both the signal and the statistical structure of the noise are completely known. For the time-varying filter this implies that the *estimated* envelopes in (3.11) are replaced by exactly *known* envelopes.

For the ensemble average and each of the four filtered averages the normalized mean-square error *(mse)* was calculated according to

$$\text{mse} = \frac{1}{K\sigma_n^2} \sum_{k=1}^{K} \{\hat{s}(k) - s(k)\}^2 \qquad (5.1)$$

where K is the number of samples used (256), σ_n^2 the noise power, and $\hat{s}(k)$ represents one of the five estimates. The respective errors are shown in the bars of Figure 7.6f.

5.2 Results

From Figure 7.6f it is evident that averaging, *a posteriori* "Wiener" filtering (a.p.w.f.), theoretical Wiener filtering, time-varying filtering (t.v.f.) and "theoretical" time-varying filtering (Figure 7.6a-e) lead, in that order, to a progressively smaller estimation error. In particular, the time-varying filter (Figure 7.6d) provides a smaller estimation error than the *optimal time-invariant theoretical* Wiener filter (Figure 7.6c). This is a general finding in si-

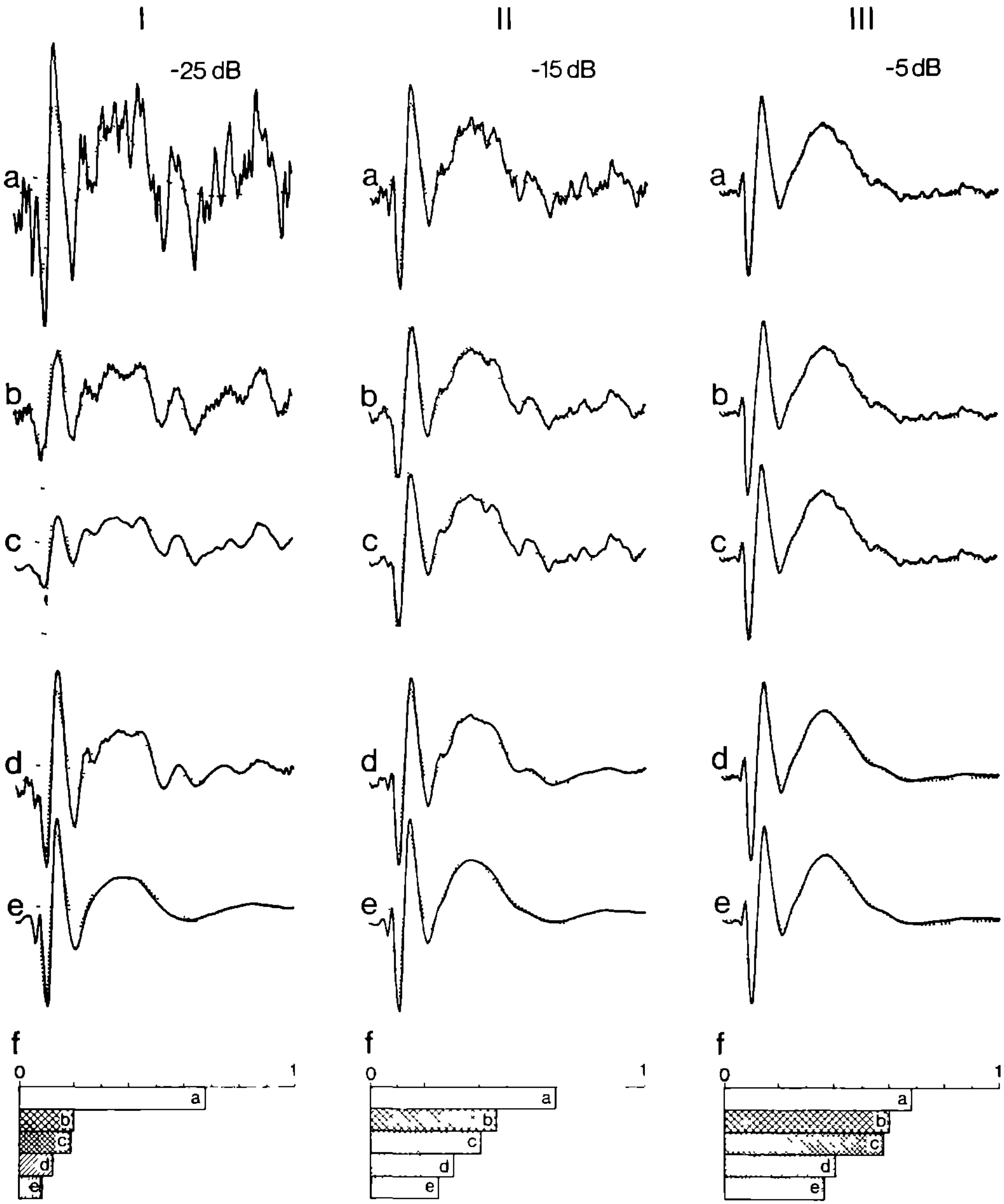

Figure 7.6I-III
Simulation example, showing a comparison between various processing techniques, on the basis of an artificial signal (dotted lines) masked by additive, filtered noise at three initial signal-to-noise ratios (SNR prior to averaging: I: -25 dB; II: - 15 dB; III: -5 dB).
a: *ensemble average of 128 sweeps;* b: a posteriori *"Wiener" filtered*

mulation studies concerning transient signals and the remarkable aspect of it is that t.v.f., without making use of any knowledge on signal and noise structure, does better than theoretical Wiener filtering, which relies entirely on this *(a priori)* knowledge. This fact provides probably the most convincing evidence that Wiener filtering is inadequate in transient signal estimation. Of course, this becomes even clearer upon comparison of theoretical Wiener filtering (Figure 7.6c) to its theoretical time-varying counterpart (Figure 7.6e).

Another interesting comparison can be made between t.v.f. and "theoretical" t.v.f. Here it can be seen what price must be paid for the lack of precise knowledge on signal and noise structure. The difference between these two estimates and between their mean square errors becomes relatively larger with decreasing signal-to-noise ratio. This is understandable from the fact that with lower signal-to-noise ratios the estimated signal and noise envelopes on which the calculated time-varying weighing function is based, become increasingly more variable and the filtering consequently less optimal.

Although it is general practice to compare estimation methods on the basis of mean-square error, this measure is often intuitively unsatisfactory for various reasons. In our present simulation example, the signal contains components of short duration which, in evoked potential practice, would have significant diagnostic meaning. However, a possibly small estimation error in these waveform components is easily obscured by the fact that the square error is averaged over the entire (much longer) observation interval. Thus the mean square error is dominated by the estimation accuracy of the large components of relatively long duration. In fact, the time-varying, or *instantaneous* square error would be a more appropriate measure than its time average. Instantaneous square error curves were computed, but have been omitted in Figure 7.6 for reasons of clarity. These curves emphasize the adaptive character of t.v.f., as opposed to a.p.w.f., in that it provides not only a better estimate of the initial "fast" signal components (through a smaller reduction in the signals' amplitude), but also of the final "slower" components (through a larger reduction in the varia-

version of a; c: *theoretical Wiener filtered version of* a; d: a posteriori time-varying filtered *version of* a; e: *theoretical time-varying filtered version of* a; f: *normalized mean-square errors (cf. eq. (5.1)) for the various processing techniques* a-e. *Note that* a posteriori *time-varying filtering* (d), *which does not make use of any* a priori *knowledge on signal and noise, performs better than theoretical Wiener filtering* (c), *which relies entirely on such knowledge* bility). This behaviour can also be seen upon a closer comparison in the Figures 7.6b and 7.6d. Another drawback of relying on the mean-square error is, that this measure does not give us direct access to the (amplitude) distortion that we impose on the underlying signal. For instance, application of a.p.w.f. generally leads to a smaller estimation error as compared to ensemble averaging. At low signal-to-noise ratios this reduction in error may be considerable, but at the same time the estimate becomes a heavily distorted replica of the underlying signal, and misleading interpretations may result (see also de Weerd and Martens, 1978), Similarly, the t.v.f. method will also break down at very low signal-to-noise ratios; in this respect the situation in Figure 7.6I may probably be regarded as a borderline case as far as the initial small signal components are concerned. At the other extreme, with a 20dB higher signal-to-noise ratio (Figure 7.6III), the average itself is already relatively noise free, and although t.v.f. still provides a significant improvement, one might rightly argue that its application does not contribute much to a better interpretation of the signal. (Note, that by visual inspection one is inclined to decide that a.p.w.f. does not give any improvement at all in this case.) However, in this region of signal-to-noise ratios t.v.f. may well be used to reduce the number of averaged sweeps. In the practice of evoked potential estimation it has thus been possible to reduce the number of stimulus presentations by factors from approximately 2 up to 10. For signal-to-noise ratios in between the two previous extremes (e.g. Figure 7.6II) t.v.f. may be used with advantage to reduce the estimation error beyond averaging and/or the number of stimulus presentations.

Of course, the above simulation provides but a single example, and the actual figures for mean-square error reductions cannot be simply generalized to arbitrary cases. However, extensive simulation studies and applications to evoked potentials have clearly demonstrated the general applicability of the t.v.f. method and its superiority to *a posteriori* "Wiener" filtering when dealing with transient signals. In this respect, the above example certainly is a typical one.

6. Concluding Comments

In this study we have dealt with the optimal estimation of a nonrandom transient signal, starting from a homogeneous ensemble consisting of identical replica of this signal in the presence of additive noise. This problem can be considered as a typical one in evoked potential estimation. Previous papers in this field have been restricted to time-*in*variant solutions such as the method of *a pos-*

teriori "Wiener" filtering. In this technique the ensemble is averaged and the result passed through an "optimal" filter which is calculated from estimated power density spectra of signal and noise. Estimation of the latter spectra from the ensemble is, in turn, achieved by using special averaging techniques.

When dealing with time-varying signals (and noise), it is unlikely that time-*in*variant systems will produce good results; an example has been given in the simulation study. As a more suitable solution we have therefore postulated a *time-varying filter* by simply replacing the estimated power density spectra of signal and noise, as they appear in the "Wiener" filter, by their time-varying counterparts.

At present we are unable to prove formally that this solution is, in the mean-square error sense, truly optimal, since we have not carried out a rigorous mathematical treatment starting from very basic assumptions. However, if we consider "theoretical" time-varying filtering, that is the filtering based on *a priori knowledge* of signal and noise structure, we conclude from extensive simulation studies that this method is likely to be close to optimal. The problem of paramount importance with respect to optimality does therefore not lie in this part of the theory, but rather in the fact that in practice we can do no better than to replace the presupposed *a priori* knowledge on signal and noise by *a posteriori* estimates. Since these estimates become less "certain" with decreasing signal-to-noise ratios, the filtering accordingly becomes less optimal.

Time-varying filtering achieves its improvement beyond averaging through a reduction in the variance of the estimated waveform, that is, by attenuating the noise components. However, if signal and noise show a *simultaneous* overlap in time *and* frequency, not only the noise components are attenuated, but the signal components as well. In other words, the *variance* of the estimated waveform is reduced, at the cost of an increased *bias* (that is the reduction of the amplitude of the signal components). This bias may become large when time-varying filtering is applied to averaged waveforms with, loosely phrased, an extremely low signal-to-noise ratio. In such cases time-varying filtering must be discouraged since the appealingly smoothed estimates which usually result may reflect a heavily distorted image of the "true" underlying evoked potential. Most obvious improvements are obtained at intermediate signal-to-noise ratios (i.e. if the waveform can be reasonably recognized in the background noise), while at high signal-to-noise ratios the method may help to reduce the number of stimulus presentations.

We have been acutely aware that the usefulness of a particular method depends in no slight measure on its ease of implementation. Therefore,we have paid particular attention to the development of a time-varying filtering package that can easily be interfaced with virtually any current software averaging program. It is hoped that a further evaluation of the time-varying filtering method, already initiated at some centres, is thereby encouraged, not only in the field of evoked potential estimation, but also in other areas of research where averaging is the method of choice, but where a further improvement is both desirable and likely.

Acknowledgement
The authors are indebted to W.L.J. Martens, who elaborated a rudimentary form of time-varying filtering in partial fulfillment of his M.S. degree in electrical engineering, in 1977. Also, the authors thank P.I.M. Johannesma, D.F. Stegeman, G.J.H. Uyen and A.J.H. Vendrik for their critical reading of the manuscript.

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Chapter 8

Application of Time-varying Filtering to Averaged Evoked Potentials

Abstract

The usefulness of *(a posteriori)* time-varying filtering for the practice of evoked potential estimation is considered. In pursuit of a suitable methodology for evaluating the method several possibilities are reviewed. It is argued that the lack of knowledge concerning the processes underlying evoked potentials impedes an objective, quantitative, evaluation. As a substitute, time-varying filtering is compared to normal averaging and time-*in*variant *"a posteriori* 'Wiener' filtering" in estimating visual, somatosensory and auditory brainstem evoked potentials. It is concluded that sensible use of the method consistently leads, unlike "Wiener" filtering, to substantial improvements beyond averaging.

1. Introduction

In the previous chapters, 6 and 7, we have introduced the principles of *(a posteriori) time-varying filtering (t.v.f.)*, along with some illustrative examples to demonstrate its practical feasibility. In this chapter we will tentatively answer the question of how useful time-varying filtering is in the actual (clinical) practice of evoked potential estimation. Such a question immediately poses the problem of establishing a suitable procedure for evaluation of the t.v.f. method and we will discuss some of the possibilities in what follows.

A first possible approach is that of *simulation*. By creating a collection *("ensemble")* of artificial evoked responses and noise with *a priori* known characteristics, various processing methods can be evaluated by comparing their results with the desired waveform. The performance of each method can be further characterized by measuring the deviation of the resulting and the desired waveform and expressing this deviation in a suitable measure, such as the widely used "mean-square error".

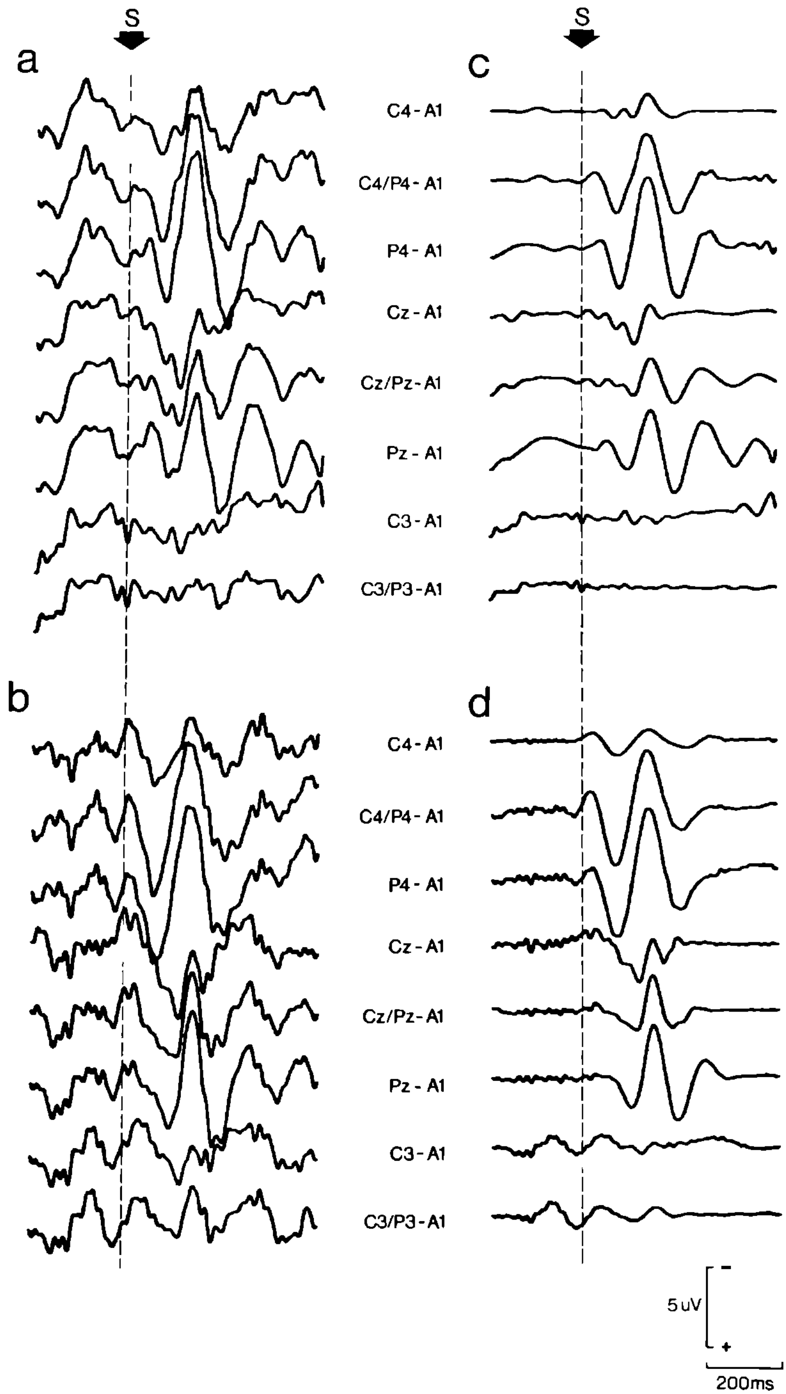
In our opinion the simulation approach is indispensable when starting a new method, in order to gain insight into its performance under *controllable* circumstances. Simulations may either be based on purely "technical" signals, such as a stylized evoked potential waveform and artificially generated additive noise (see e.g. chapter 7, section 5), or on more realistic data such as an averaged evoked potential added to spontaneous (i.e. non-stimulated) EEG.

Although in such simulations time-varying filtering has invariably demonstrated remarkable improvements beyond averaging, this fact *per se* does not prove the reliability of the method if applied to *real* evoked potential data. The major difficulty in using simulations is the necessity of making several, unverified, *simplifying assumptions*. Usually, evoked potential simulations depart, for example, from the basic assumptions of an invariant signal (representing the evoked potential waveform) and additive noise (representing the background activity). These are also exactly the assumptions underlying both the averaging and the t.v.f. technique, so that these methods obviously operate under idealized circumstances in such simulations. On the other hand, it is well known that evoked potentials, especially the ones of cortical origin, are only roughly approximated by the simple signal-plus-noise model postulated above. Actually it will be shown in section 3, for instance, that evoked potentials may exhibit a large variability, rather than being invariant.

Quite the opposite approach to the evaluation of time-varying filtering is its straightforward application in *clinical practice*, particularly to those cases where conventional averaging does not produce well-interpretable results. Naturally, under these circumstances the only possibility for evaluation lies in comparing t.v.f. results to those of other processing techniques, the aforementioned averaging technique in particular. Such a comparison could, for instance, be based on a visual assessment of the degree to which various waveform components can be discerned and recognized.

As a matter of fact, this method of evaluation has been applied over the past few years to a large number of averaged and subsequently time-varying filtered clinical evoked potentials, particularly of the somatosensory type. To simplify the evaluation the following additional techniques were employed to establish the significance and reliability of individual waveform components: (1) simultaneous measurement of eight derivations to obtain additional (topographical) information, (2) inclusion of a pre-stimulus interval (i.e. an analysis interval prior to the application of the stimulus) to obtain an impression of the remaining background activity as compared to the stimulus induced activity and (3) repetition of the measurement to assess the reproducibility of the various waveform components.

Figure 8.1 gives an example from practice showing results for a somatosensory evoked potential. Eight derivations were acquired in two subsequent sessions (for an explanation of the indicated electrode localizations, see the


S
a
C4 - A1
C4/P4 - A1
P4 - A1
Cz - A1
Cz/Pz - A1
Pz - A1
C3 - A1
C3/P3 - A1
S
c
b
d
5 uV
200ms

appendix, Figure 8.6). Parts *a* and *b* show the normal averages; parts *c* and *d* their corresponding time-varying filtered versions. The most prominent aspect in these recordings is probably the fact that all *pre*-stimulus intervals of the normal averages contain large fluctuations, apparently due to the background activity, which are to a large extent *suppressed* by time-varying filtering. (It should perhaps be emphasized that in the time-varying filtering method no use is made of the moment of stimulation). On the other hand, the *post*-stimulus intervals of the normal averages show fluctuations which are on time-varying filtering clearly *enhanced* relative to the background activity, in spite of a slight reduction in their amplitude. These fluctuations should, in all likelihood, be ascribed to stimulus induced activity; on comparison of Figures 8.1*c* and *d* it is seen that the greater part is reproduced, although not in detail.

The above example, which is only one of many, may inspire confidence in those for whom interpretation of evoked potential recordings is daily practice, but in others it may rightly leave a feeling of dissatisfaction.
The weakness in the evaluation methodology outlined above naturally lies in the fact that the "real" underlying evoked potential waveform is unknown. This makes it difficult, if not impossible, to estimate the significance of the enhancement or suppression of specific waveform components. In other words, what is actually lacking is an evoked potential *standard* to which the results of various processing methods can be objectively compared.

Some authors have investigated the problem of whether or not a normal *standard visual* (Cigánek, 1969) or *somatosensory* (Ikuta et al., 1980) *evoked potential* exists across individuals. It appears that such standards do exist, but the large interindividual variability about these standards prohibits their use as a means for comparing the performance of processing methods in

Figure 8.1a-d
Comparison of normal averages *and* (a posteriori) time-varying filtered averages *of somatosensory evoked potentials from a 7 year old patient (male) in response to electrical stimulation of the left index finger (intensity 8mA, duration 0.1 ms).* S *indicates the moment of stimulation; traces to the left hand side of this instant represent pre-stimulus intervals.* a: *normal averages (128 responses) of the derivations shown at right hand side (cf. Figure 8.6);* b: *normal averages (128 responses) of a repeated measurement in order to test reproducibility of the various waveform components;* c: time-varying filtered *versions of* a; d: time-varying filtered *versions of* b

individual cases. Apparently the availability of *intra-individual standards* is what is actually needed.

This brings us to still another method of evaluation practiced by several authors in assessing the usefulness of "*a posteriori* 'Wiener' filtering" (Hartwell and Erwin, 1976; Doyle, 1977; Kearney, 1979; Carlton and Katz, 1980). In this method an individual evoked potential standard is created by making up the average over a (very) large number of responses. This standard is then used for comparing the results of various processing techniques applied to a (much) smaller number of responses.

However, this approach gives rise to the following comments. In the first place it is required that the average over the large number of responses (which we shall denote by "*full average*") yields a reliable and reproducible waveform. Such a requirement can nearly always be met in healthy subjects, but much less often in patients. Secondly, it must be kept in mind that by accepting the full average as a *standard*, i.e., an adequate estimate of some "true", underlying evoked potential, it is implicitly assumed that averaging is a *perfect* estimation technique, which is only the case under idealized circumstances. Specifically, the full average may have been distorted to some extent by both a short term and a long term latency and amplitude variability in individual responses; this is known to occur particularly in cortical evoked potentials.

The foregoing discussion illustrates that an adequate methodology for evaluating time-varying filtering is by no means self evident. In conclusion, it appears to be unavoidable that the evaluation problem is tackled in a more qualitative manner, using plausibility arguments, instead of persisting in a quantitative, but obviously impossible, analysis.

2. Actual Evaluation Method

By continuing the above lines of thought and combining some of the previous ideas, we come to the following set-up for analyzing the performance of time-varying filtering (Figures 8.2-8.5).

1) In a group of healthy subjects evoked potentials from the visual, somatosensory, and auditory modality are obtained using a *large number* of stimulus presentations. Only those response ensembles yielding reproducible, almost "noise-free" averaged evoked potentials *("full averages")* are considered for further analysis. Reckoning with the possibility that the full average might have been distorted, due to the aforementioned variability in latency and amplitude, this averaged waveform is assumed to represent no more than a global replica of the "standard" (underlying) evoked potential.

2) Next, the full ensemble is split up into several (4-16) subsequent, non-overlapping, subensembles of smaller size. This size is chosen on the basis of a visual estimate of the "signal-to-noise ratio" in these subensembles. Four of these ensembles, judged to be representative for the remaining ones, are independently averaged and their resulting waveforms presented in a superimposed fashion. No more than four subensembles are shown to prevent blurring of the individual waveforms.

3) To the five averages thus obtained (namely the *full average* and the four *subensemble averages)* time-varying filtering is applied, each filter being based, of course, on its own ensemble. The t.v.f. results of the four subensembles are again superimposed to allow an estimation of the consistency and variability of the filtered waveforms.

4) As a further comparison, the results of the quite popular, but controversial *"a posteriori* 'Wiener' filtering" *(a.p.w.f.)*, applied to the same five averages, are presented in the same manner. A comparison of the results of time-varying filtering to those of the time-*in*variant "Wiener" filtering can reveal the relevance of taking the time-varying structure of the evoked potential waveforms into account.

As far as the *results* are concerned, we have deliberately chosen to present only a few examples in detail, instead of overburdening the reader with a surplus of different waveforms. We make no claim that these examples are typical for the broad line of clinical applications, although we believe that they well represent the merits and pitfalls of each of the methods. In each example we will further concentrate on one derivation only, namely the one most commonly used with respect to the modality under consideration. Firstly, the application of the various methods to cortical visual and somatosensory evoked potentials will be considered, and then spinal and brainstem auditory evoked potentials will be dealt with in a subsequent section. Details of the stimulation and recording procedure are thought to be of lesser importance in the present context and can be found in the appendix.

3. Application to Cortical Evoked Potentials

In Figure 8.2 the results for a normal *visual* evoked potential, elicited by a bright flash applied to the right eye, have been presented. Part *a* shows the full average (128 responses), *b* the time-varying filtered version and *c* the "Wiener" filtered version of this average. The main components of the evoked potential have been indicated, in conformity with Cigânek (1969), by Roman numerals. The labeling of these components serves no other goal than to

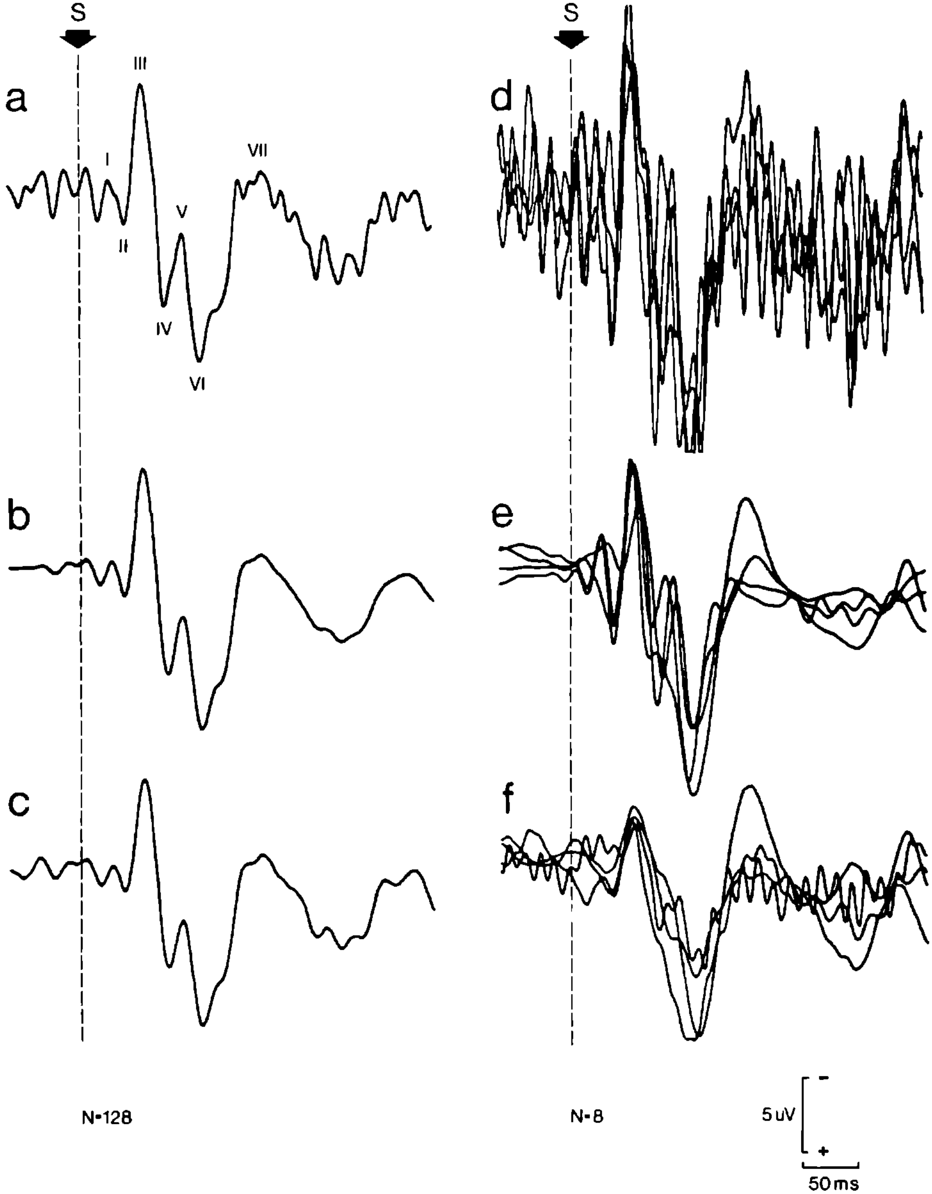


Figure 8.2a-f

Results of applying various processing methods to a visual *evoked potential from a healthy male (21 years), recorded from 01-M1, in response to a bright flash applied to the right eye.* S *indicates the moment of stimulation. Calibration bars apply equally to all waveforms.* Left half figure: a: *normal average of 128 responses;* b: (a provide the reader, unfamiliar with this type of evoked potential, with an easy reference. The latencies of all components fall well within the normal range (Cigánek, 1969).

The pre-stimulus interval of the full average (Figure 8.2a) still shows relatively large fluctuations, so that it must be concluded that the background activity has been insufficiently averaged out. Time-varying filtering (Figure 8.2b) eliminates this activity to a large extent, both in the prestimulus interval and in the tail of the response (beyond wave VII). "Wiener" filtering (Figure 8.2c) shows a similar, though less pronounced result.

The differences between the three processing methods become more clear when results of the subensembles (of 8 responses each) are considered. The superimposed subensemble *averages* (Figure 8.2d) reveal only components III and VI; the others remain obscured by the background activity. On the contrary, all components are readily recognized in the *t.v.f. averages* (Figure 8.2e). Apparently, components II and III are the most stable ones; the others show considerable variability, or are even incidentally absent. The latency variability of the various components appears to be in good agreement with the normative data given by Cigánek (1969).

The *a.p.w.f. averages* (Figure 8.2f) obviously show a reduced background activity, so that components III and VI become relatively more enhanced, albeit with reduced amplitude. However, the whole waveform is rather distorted as can be seen from the absence of components I, IV and V.

Figure 8.3 shows a *somatosensory* evoked potential, elicited by electrical stimulation of the right index finger. The presentation is the same as in Figure 8.2. The nomenclature of the waveform components is given in accordance with the current "polarity-latency" convention (see e.g. Lehmann and Callaway, 1979 and Barber, 1980). This system identifies various components by their polarity (*N*egative or *P*ositive) and (normal) latency in milliseconds. In the present example it will be of further interest to note that the cortical somatosensory evoked potential is generally divided into a *specific* complex

posteriori) time-varying filtered (t.v.f.) *version of* a; c: a posteriori *'Wiener' filtered* (a.p.w.f.) *version of* a. Right half figure: *same methods applied to* subsets *of the ensemble on which the average in* a *is based, namely* d: *superimposition of four subensemble averages (*8 *responses each);* e: *corresponding* t.v.f. *versions of* d; f: *corresponding a.p.w.f. versions of* d

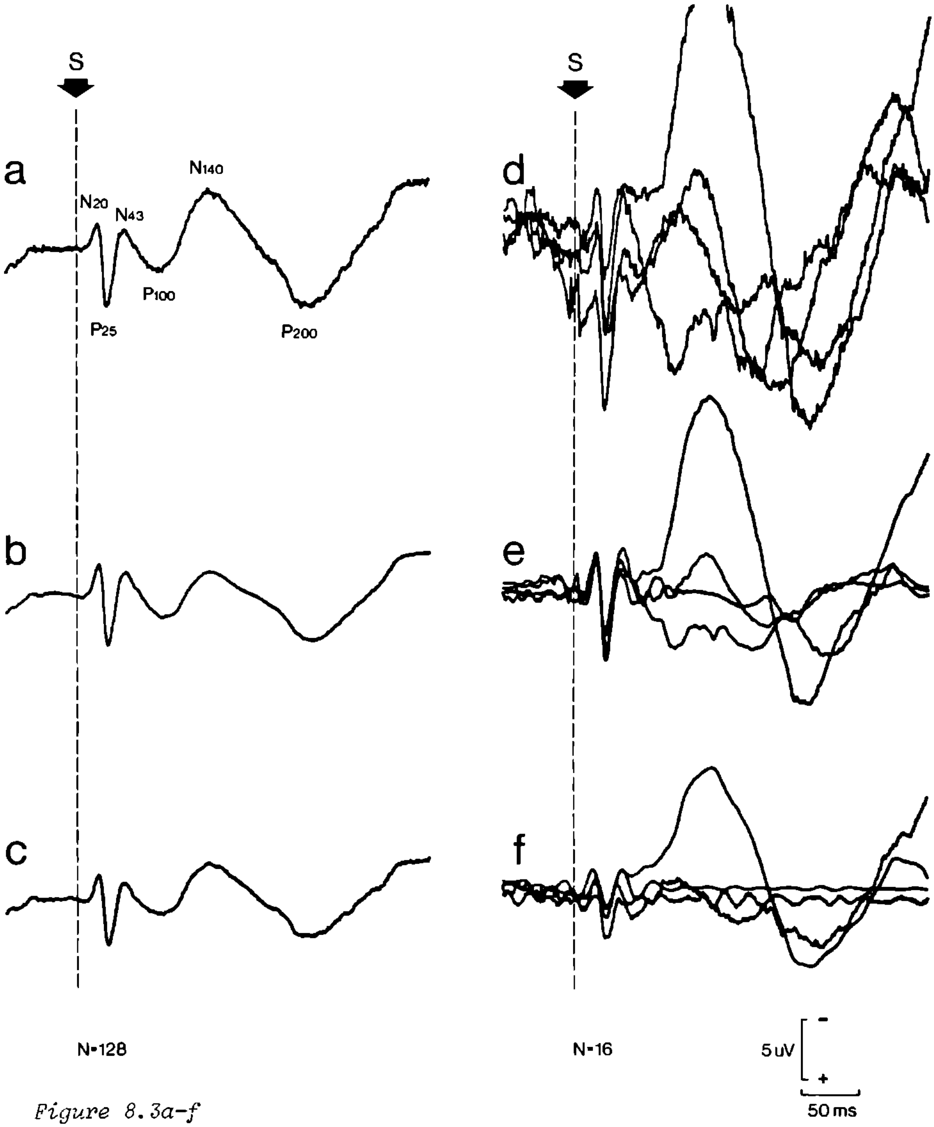


Figure 8.3a-f
Results of applying various processing methods to a somatosensory *evoked potential from a 4 year old boy, recorded from C3/P3-A2 in response to electrical stimulation of the right index finger (intensity 10mA, duration 0.1 ms).* a: *normal average of 128 responses;* b: t.v.f. *version of* a; c: *a.p.w.f. version of* a; d: *superimposition of four subensemble averages (16 responses each);* e: *corresponding* t.v.f. *versions of* d; f: *corresponding a.p.w.f. versions of* d. *Further details as in Figure 8.2*

(with a latency up to approximately 50 ms) and a nonspecific complex (with a latency beyond 50ms). The specific complex is thought to be a local response from the specific somatosensory projection area, while the nonspecific complex can be considered as a diffuse cortical response.

From Figure 8.3a it is seen that the full average is almost noise-free so that time-varying filtering (Figure 8.3b) and "Wiener" filtering (Figure 8.3c) of this average have little effect, except for a remarkable amplitude reduction in the "late" components N140 and P200. On inspection of the superimposed subensemble *averages* (Figure 8.3d) the reason for this suppression becomes obvious. Apparently, the amplitude of the late components in the *full average* is, to a large extent, determined by the casual occurrence of high amplitude responses. Both time-varying and "Wiener" filtering interpret occasional phenomena as belonging to the "noise", so that their amplitude will be further reduced.

More generally, it can be stated that amplitude reduction on application of t.v.f. in an otherwise noise-free averaged evoked potential waveform always indicates a large variability and/or the presence of outliers in individual responses. Although it cannot be entirely ruled out that the actual large amplitudes are due to artefacts, it is much more likely that these reflect true variability in the nonspecific complex of the underlying evoked potential. This is, in particular, suggested by the fact that the specific complex (N20-P25-N43) is quite well reproduced in all subensemble averages (Figure 8.3d).

The *t.v.f. averages* (Figure 8.3e) greatly enhance the visibility of the specific complex and accentuate the contrast between the stability of this complex and the large variability of the nonspecific one. On the contrary, *"Wiener" filtering* (Figure 8.3f) does not seem to contribute at all to an improved estimation. The method not only reduces the background activity, but the evoked potential components with it. It appears to be no overstatement to conclude that in this case the results have been worsened, rather than improved.

4. Application to Spinal and Brainstem Evoked Potentials

Contrary to the evoked potentials of cortical origin, spinal and brainstem evoked potentials are relatively stable responses which can be assumed to have little or no interaction with the background activity. As such, these potentials comply better with the signal-plus-noise model discussed in section 1. Consequently we might state that possible variability, observed in the results of superimposed subensembles, is more likely to be due to remaining background activity than to inherent variability in the evoked potential itself. This ten-

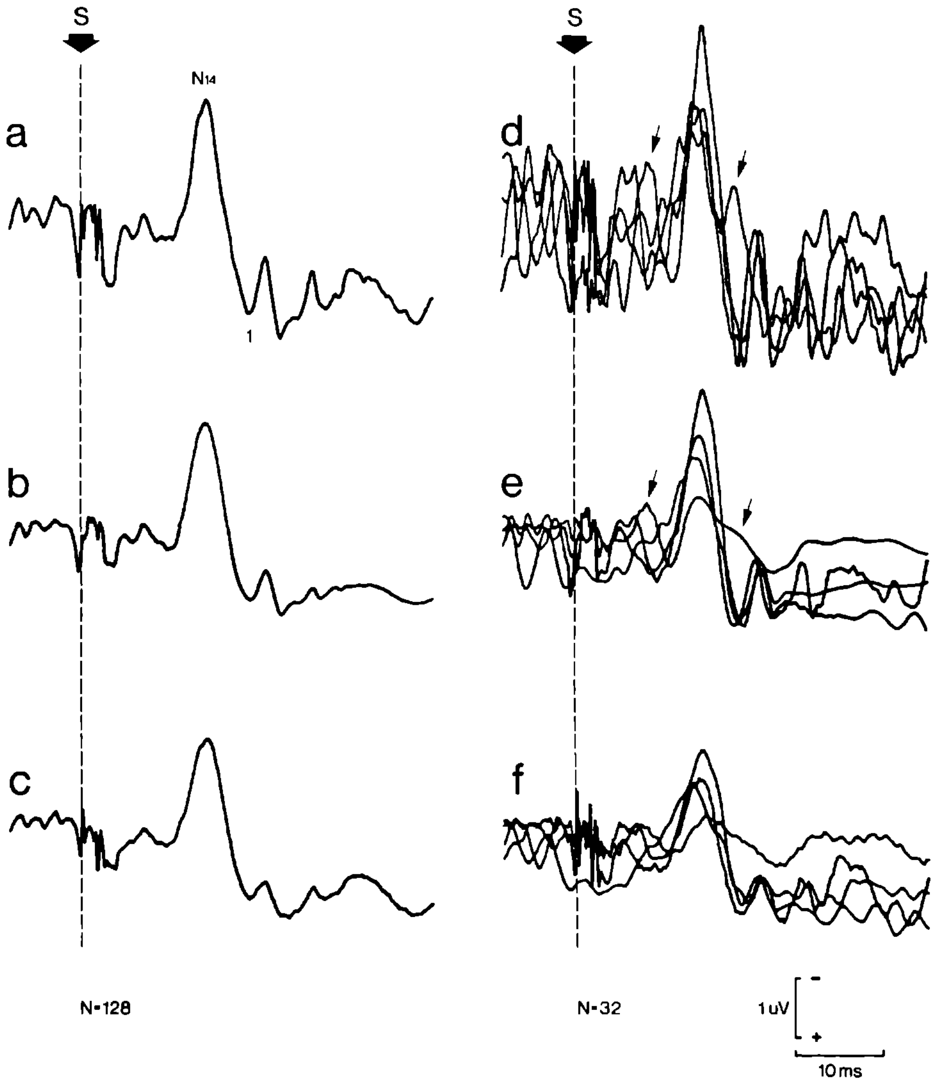


Figure 8.4a-f
Results of applying various processing methods to a somatosensory *evoked potential recorded from the neck (C7-Fz) of a healthy female (25 years) in response to electrical stimulation of the left median nerve at the wrist (intensity 10mA, duration 0.1 ms).* a: *normal average of 128 responses;* b: t.v.f. *version of* a; c: *a.p.w.f. version of* a; d: *superimposition of four subensemble averages* (32 *responses each);* e: *corresponding* t.v.f. *versions of* d; f: *corresponding a.p.w.f. versions of* d. *Further details as in Figure 8.2*

tative assertion is important for a proper interpretation of the results that follow.

Figure 8.4a shows the full average (128 responses) of a *somatosensory* evoked potential, recorded from the *neck* on stimulation of the left median nerve at the wrist [1]. The component labeled N14 (according to the polarity-latency convention, see section 3) is thought to reflect the cervical cord afferent volley (Cracco, 1973), while the component labeled "1" probably reflects the initial activity of the somatosensory cortex and is normally of opposite polarity ("N20") in scalp recordings (cf. Figure 8.3a). The waveform in Figure 8.4a suggests the presence of still later components beyond the latter one, the significance of which is, however, unclear. Figure 8.4b shows that these later "components" are largely suppressed by time-varying filtering which may be due, as in the previous example, to their possibly large variability. "Wiener" filtering of the full average (Figure 8.4c) shows approximately the same result.

The superimposed subensemble *averages* are shown in Figure 8.4d. Although in these averages the components N14 and "1" can still be distinguished, they are clearly enhanced by *time-varying filtering* (Figure 8.4e). However, one of the time-varying filtered subensemble averages (indicated by arrows) does not resemble the waveform of the full average (Figure 8.4a) at all. On a closer inspection of the corresponding subensemble average (Figure 8.4d, also indicated by arrows) it appears that this waveform not only deviates from the full average, but also shows large random fluctuations, which may perhaps be ascribed to muscle artefacts.

Let us suppose now that this particular subensemble were the only one available for analysis. The noisy character of the subensemble *average* would probably have led us to reject that average as being unreliable. On the other hand, the *smooth* waveform of the corresponding *t.v.f. average*, suggesting reliability, might easily have led to erroneous interpretations. This is a dangerous aspect of the t.v.f. method, to which we will come back in the discussion.

The *a.p.w.f. averages* (Figure 8.4f) demonstrate an enhanced visibility of the two main components, although their amplitude has been significantly reduced. Unlike the previous examples it appears that for the present case the difference between "Wiener" and time-varying filtering is less pronounced.

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[1]) The author thanks Mr. H.T.M. Haenen, Department of Clinical Neurophysiology, University of Groningen, for providing these data.

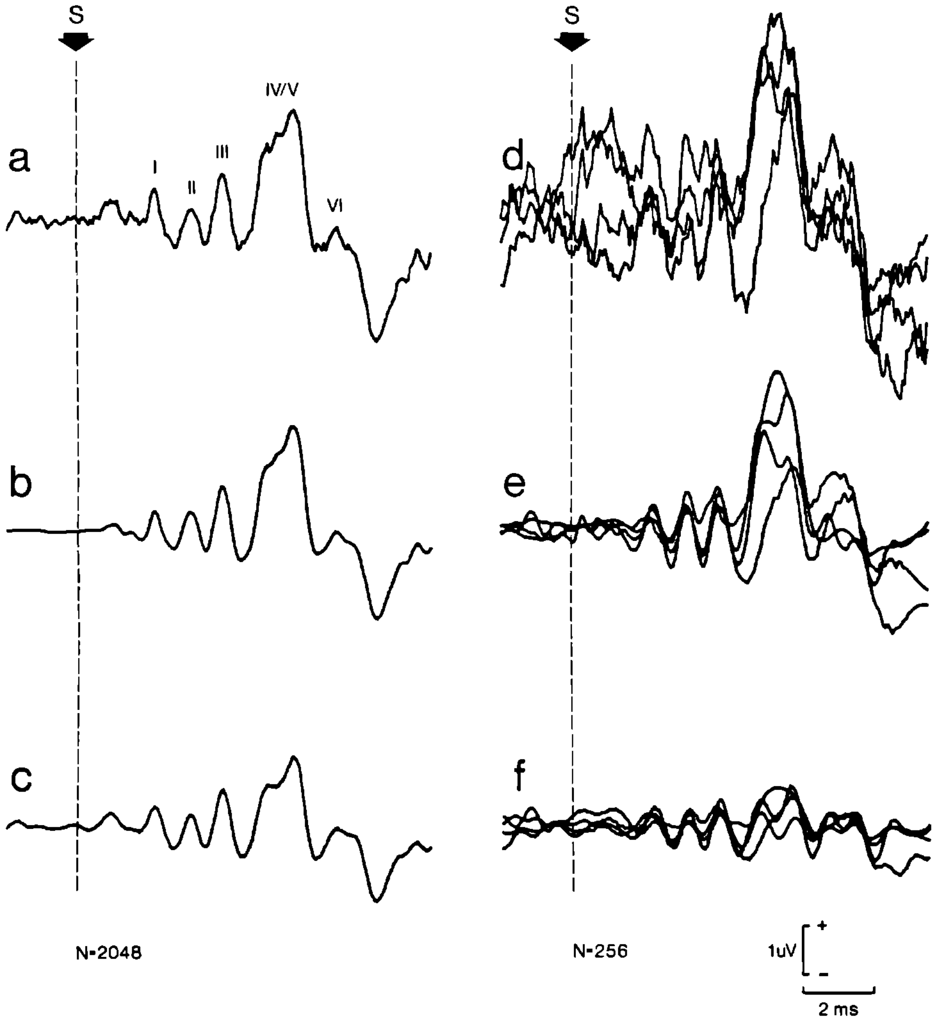

Figure 8.5a-f

Results of applying various processing methods to a brainstem auditory *evoked potential (Cz-M1) from a healthy female (24 years) in response to binaurally applied 60 dB SL rarefaction clicks.* a: *normal average of 2048 responses;* b: t.v.f. *version of* a; c: *a.p.w.f. version of* a; d: *superimposition of four subensemble averages (256 responses each);* e: *corresponding* t.v.f. *versions of* d; f: *corresponding a.p.w.f. versions of* d. *Further details as in Figure 8.2*

Figure 8.5 shows a *brainstem auditory* evoked potential after binaural stimulation with 60 dB clicks. In the full average (Figure 8.5a) a sequence of several waves, occurring within the first 10 ms after the stimulus, is seen. In accordance with the notation of Jewett and Williston (1971) the components are labeled with Roman numerals. An interesting aspect of this evoked potential is that it has been fairly well established from which anatomical regions the various components arise (see e.g. Buchwald and Huang, 1975).

Time-varying filtering (Figure 8.5b) apparently removes some remaining noise from the full average and so does "Wiener" filtering (Figure 8.5c) albeit to a lesser extent. However, "Wiener" filtering also causes a significant reduction in the IV/V components.

More distinct differences between the methods are seen in the results of the various subensembles (of 256 responses each). The superimposed subensemble *averages* (Figure 8.5d) only reveal component V and perhaps I. Recognition of the other components is difficult, if not impossible, due to the large background activity. *Time-varying filtering* (Figure 8.5e) produces considerable improvements: all components, except perhaps wave VI, are readily identified, while their amplitudes are also well within the range of those of the full average (Figure 8.5a). It is interesting to compare these results to the *a.p.w.f. averages* (Figure 8.5f). Undoubtedly, in the latter waveforms the variability of the background activity has been reduced to such an extent that all components are easily recognized. Yet, "Wiener" filtering has seriously distorted the waveform, components IV/V in particular.

5. Discussion

The need for improved estimation methods in evoked potential analysis stems from the fact that, in clinical practice, ensemble averaging leads as often as not to noninterpretable waveforms with an apparently insufficiently reduced background activity. In many such cases a further increase in the number of averaged responses, and thereby in stimulus presentations, may be either practically impossible (e.g. when investigating young infants), or ineffective. The latter situation frequently occurs in prolonged recording sessions, where the state of the subject is likely to change which may cause a degradation of the averaging process. More generally, the development of improved methods is motivated by the desire to minimize both the burden for the patient and the ambiguity in recognizing and evaluating evoked potential waveforms and their individual components.

The present chapter aimed at demonstrating the usefulness of *(a posteri-*

ori) time-varying filtering in this respect. A major problem which we had to deal with is how a suitable methodology for evaluation of this method on *real* evoked potential data should be established. Strictly speaking, a thorough evaluation is impossible without a proper evoked potential model, based on detailed knowledge of the processes underlying evoked potentials. The lack of such knowledge, especially with respect to cortical evoked potentials, means that we could do no better than to rely on a qualitative approach, using intuition and experience. On this background, we have illustrated the application of time-varying filtering to evoked potentials of the visual, somatosensory and auditory modalities by comparing the method to normal averaging and "*a posteriori* 'Wiener' filtering".

As far as the latter comparison is concerned, the above examples have clearly illustrated the superiority of time-varying filtering in estimating evoked potentials. Although it appeared that "Wiener" filtering occasionally led to an enhanced visibility of individual waveform components, the waveform as a whole usually became seriously distorted.

On the other hand time-varying filtering proved, in general, to be a more reliable method for improving the evoked potential waveform beyond averaging or, alternatively, for reducing the number of averaged responses while retaining about the same quality of the evoked potential waveform. In clinical practice it has thus been possible, using time-varying filtering, to unequivocally identify and measure evoked potential components and their latency from patients in which the normal average led to ambiguous results. In less problematic cases the number of stimulus presentations could generally be reduced by factors of between approximately 2 and 10.

Even so, we prefer to advocate the use of time-varying filtering *for improving the averaged waveform*, rather than for reducing the number of stimuli. The reason for this preference is that time-varying filtering achieves its improvement beyond averaging by an adaptive filtering which affects, to some extent, the evoked potential waveform as well [1)]. The filtering may result in some amplitude reduction in the various waveform components, so that *amplitude* measures (unlike *latency* measures) become less reliable, particularly when the remaining background activity is comparatively large and the filtering accordingly severe.

The above argument also explains why the *blind application* of time-vary-

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1) The precise nature of this influence has been explained in the preceeding chapter (section 6)

ing filtering, like many other sophisticated methods, carries the risk of serious misinterpretations (cf. section 4). These can easily be avoided if the following basic rules are taken into account: (1) time-varying filtered waveforms should primarily be considered as *supplements*, not as *replacements* for normal averages; (2) the method should not be applied to evoked potentials that are still totally buried in the background activity and (3) as a rule, one should repeat the evoked potential measurement to assess the reproducibility of the resulting waveforms. In clinical practice we have often found it a useful procedure to acquire an ensemble of a normal number of responses, which is then averaged and also split up into several (usually two to four) subensembles. Comparison of time-varying filtered subensemble averages (e.g. by superimposition) then allows an assessment of the variability, while a comparison to the normal average over the entire ensemble may prevent misleading, or perhaps too speculative, interpretations.

Appendix: Stimulation, Recording and Analysis Procedures

A1. General Set-up

All evoked potentials were obtained using standard clinical procedures. Subjects were comfortably seated on a chair in a semi-darkened room with ambient temperature between 22 and 24° C. Recordings were made with Ag/AgCl electrodes, attached with collodion and filled with conductive jelly. In scalp recordings, the international 10-20 system for electrode localization was used (Figure 8.6) except that for somatosensory evoked potentials additional electrodes were placed halfway between standard positions (dotted circles in Figure 8.6). For spinal evoked potential recordings an additional electrode was placed over the 7th cervical vertebra (C7). Electrode-impedance was measured and maintained below 1-2 kΩ. Recordings were amplified using either a Siemens EEG machine (Elema-Schönander Mingograf 16) with the filters bypassed, or a Nicolet HGA 200A amplifier system. The amplifiers were connected to a programmable 4-pole Bessel analog filter set with artefact detection and signaling (Difa MU 16), which was in turn connected to a 10 bit AD converter (AR11, Digital Equipment Corporation). Only those individual responses not contaminated with artefacts were sampled, processed and stored on disk (PDP 11/34 with dual RK05, Digital Equipment Corporation). The entire system was periodically calibrated by applying a 10 μV sinewave to the electrode terminator box and computing the mean power, after 16 periods of the sinewave had been averaged to reduce possible effects of noise interference. Parallel to the digital processing, all measurements were recorded on analog tape (Philips EL1016, 14 channel instrumentation

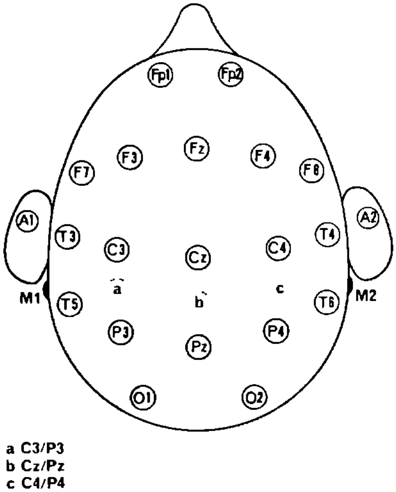

Figure 8.6
Electrode placements and nomenclature for scalp recordings, according to the international 10-20 system (Jasper, 1958). Head seen from above (nose up). Additional positions for recording over the somatosensory projection area (for median nerve stimulation) have been shown by dotted circles. M1 and M2 refer to left and right mastoid respectively

recorder). The entire recording and stimulation procedure was under computer control, timing pulses to external stimulators being delivered via a general I/O interface (DR11-C, Digital Equipment Corporation) and a (laboratory made) isolation unit. All recordings included a pre-stimulus interval to a size of 25% of the total time base.

Prior to further processing, stimulus artefacts in individual responses were removed, if necessary, by replacing 4 to 8 samples just after the moment of stimulation by samples randomly taken from the pre-stimulus interval. Averaging of the acquired responses was carried out using standard procedures. Time-varying filtered averages were obtained using a standard time-varying filtering software package (see chapter 7, section 4). "Wiener" filtered averages were obtained using a similar package, according to the procedures described in chapters 2 and 3. To avoid the problem of periodic convolution in time-varying and "Wiener" filtered waveforms, a quarter of the total time base (equally divided over the pre- and post-stimulus interval) was discarded after completion of the respective filtering procedures (see chapter 7, section 3.3 for further discussion on this point). Results were presented on a Tektronix 4010 terminal with hard copy unit (Tektronix 4601).

A2. Modality Specific Procedures

Visual evoked potentials were elicited using a white flash of 20 μs duration and an intensity corresponding with an electrical energy of 0.5 Joules. Subjects were sitting in front of a Xenon lamp (Siemens EM 720) at a distance of 25 cm with eyes open. Flashes were presented *at random* with a stimulus interval ranging from 3 to 5 s (average 4 s). Recordings were made with a bandpass of 0.1 Hz - 400 Hz (-3 dB points), and sampled at a rate of 1 kHz using 512 data points.

Somatosensory evoked potentials were elicited by brief electric shocks of 0.1 ms duration and a usual intensity of 10 mA, delivered by a Nicolet NIC 1003 or DISA 1500 system. Stimuli were presented at a regular rate of 11 Hz for spinal evoked potentials, and *at random* with an interval ranging from 3 to 5 s (average 4 s) for cortical evoked potentials. Spinal evoked potentials were recorded with a bandpass of 10-800 Hz, and sampled at a rate of 4 kHz using 256 data points. Cortical evoked potentials were recorded with a bandpass of 0.1-400 Hz, and sampled at a rate of 1 kHz using 512 data points.

Auditory brainstem evoked potentials were elicited by binaural clicks of 0.1 ms duration and an intensity of 60 dB SL, delivered by a TDH-39 headphone set and a Nicolet NIC 1001 stimulator. Only rarefaction clicks were used at a regular rate of 11 Hz. Recordings were made with a bandpass of 30 Hz-3 kHz, and sampled at a rate of 16 kHz using 256 data points.

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Samenvatting en Conclusies

1. Inleiding, Probleemstelling en Resultaten

(Hoofdstukken 1 en 8)

Evoked potentials ("opgewekte potentialen") zijn electrische spanningsveranderingen die kunnen worden afgeleid van delen van het perifere en centrale zenuwstelsel na stimulatie van een van de zintuigsystemen. Het meest onderzocht zijn evoked potentials van de auditieve, somatosensorische en visuele modaliteit. Voor het opwekken ervan wordt doorgaans gebruik gemaakt van kortdurende stimuli, b.v. een akoestische klik, een korte electrische schok of een lichtflits.

Vooral gedurende de laatste tien jaar heeft het registreren en analyseren van evoked potentials een steeds vooraanstaander plaats ingenomen in het geheel van klinisch electrofysiologische onderzoeksmethoden bij de mens. Belangrijke voordelen van dit type onderzoek zijn het niet-invasieve karakter - er kan worden volstaan met registratie (en stimulatie) door middel van oppervlakte-electroden - terwijl het onderzoek willekeurig vaak kan worden herhaald en niet, of slechts in geringe mate, belastend is voor de patient.

Het belangrijkste probleem dat zich bij het meten van evoked potentials voordoet is dat deze potentialen meestal aanzienlijk kleiner zijn dan de "achtergrondactiviteit", d.i. alle electrische activiteit die geen relatie met de stimulus heeft. Bij registratie vanaf de hoofdhuid bijvoorbeeld, vormt de immer aanwezige spontane cerebrale electrische activiteit (EEG) met amplitudes van 50 tot 100 µV de belangrijkste stoorbron voor evoked potentials, die qua amplitude uiteenlopen van 0.1 tot ca. 30 µV. In het algemeen is het dus onmogelijk om evoked potentials rechtstreeks waar te nemen.

De gebruikelijke oplossing van dit probleem is de toepassing van de zgn. middelings-("averaging") techniek. Hierbij geeft men de te onderzoeken persoon een (groot) aantal identieke stimuli waarna de verkregen individuele responsies m.b.v. een computer of speciaal daarvoor geschikt apparaat ("averager") worden gemiddeld. Uitgaande van de veronderstelling dat de evoked potential (het "*signaal*") in individuele responsies steeds identiek is, zal deze na middeling onveranderd blijven, terwijl de achtergrondactiviteit, die geen relatie met de stimulus heeft en als "*ruis*" kan worden opgevat, op den duur zal worden uitgemiddeld. Onder een aantal fysische voorwaarden is de verbetering van de "*signaal-ruis verhouding*" evenredig met de wortel uit het aantal malen dat men middelt.

Gezien de geringe amplitude van evoked potentials ten opzichte van de achtergrondactiviteit is het veelal noodzakelijk een groot aantal stimuli te geven (varierend van enkele tientallen tot enkele duizenden). Omdat deze stimuli met een bepaalde minimum intervaltijd (die afhankelijk is van het type evoked potential) moeten worden aangeboden, ligt de onderzoeksduur al snel in de ordegrootte van zo'n 5 à 10 minuten.

Anderzijds is het wenselijk, en soms noodzakelijk, de onderzoeksduur aanzienlijk te bekorten. Op de eerste plaats wordt dit ingegeven door het streven de patient zo weinig mogelijk te belasten. Bij het onderzoeken van kinderen, die over het algemeen minder geduld hebben dan volwassenen, is een korte onderzoeksduur vaak zelfs noodzaak. Op de tweede plaats speelt er ook een meer fundamenteel probleem. De aanname dat de evoked potential in successief verkregen responsies steeds identiek is blijkt voor corticale evoked potentials slechts een ruwe benadering, die met name bij langdurige registraties niet meer opgaat. Bovendien verandert in dat geval doorgaans het karakter van de achtergrondactiviteit, hetgeen de middelingsprocedure nadelig beinvloedt. In de klinische praktijk heeft het dan ook meestal geen zin om, bij moeilijk opwekbare evoked potentials, langdurig door te blijven middelen.

De bovenomschreven problematiek heeft in het verleden geleid tot twee verschillende onderzoekslijnen. De eerste lijn houdt zich bezig met het ontwikkelen van alternatieve schattingsmethoden, uitgaande van de gedachte dat mogelijkerwijze aan één of meer voorwaarden, die aan de middelingstechniek ten grondslag liggen, *niet* is voldaan.

De tweede lijn van onderzoek, waarbinnen ook dit proefschrift valt, houdt zich bezig met de vraag of er betere schattingsmethoden dan de middelingsprocedure denkbaar zijn, aannemende dat *wel* aan alle voorwaarden van laatstgenoemde techniek is voldaan.

In dit proefschrift wordt zowel vanuit de theorie als vanuit de praktijk aannemelijk gemaakt dat dergelijke methoden inderdaad bestaan. In concreto wordt de in de literatuur reeds langer bekende, maar controversiële *"a posteriori* 'Wiener' filtering" techniek gegeneraliseerd tot een *(a posteriori) tijd-variante* filter methode. Deze laatste methode is gebaseerd op een dynamische, tijd-variante "filtering" van de signaalvorm, verkregen *na* middeling van individuele responsies (vandaar de term *"a posteriori"*). Het filter zelf wordt, voor iedere gemiddelde evoked potential afzonderlijk, met behulp van speciale middelingstechnieken uit de verzameling van individuele responsies berekend.

De toepassing van de tijd-variante filter methode op zowel gesimuleerde, als op klinisch gemeten auditieve, somatosensorische en visuele evoked potentials, laat zien dat hiermee significante verbeteringen ten opzichte van de normale middelingsprocedure mogelijk zijn. Hoewel een kwantitatieve evaluatie voor werkelijk *gemeten* evoked potentials niet goed mogelijk is, - de "echte" onderliggende evoked potential is hier immers niet bekend - blijkt in het algemeen het aantal stimuli met een factor 2 tot 10 te kunnen worden teruggebracht, óf, bij gelijkblijvend aantal stimuli, de signaal-ruis verhouding van de gemiddelde evoked potential in dezelfde orde van grootte te kunnen worden verbeterd.

Evoked potential onderzoek kan hierdoor betrouwbaarder en sneller, dus minder belastend voor de patient, worden uitgevoerd. Daarbij moet worden opgemerkt, dat een meer wijdverbreide klinische toepassing van de tijd-variante filter methode, door de nog steeds dalende prijs/prestatie verhouding van microprocessors en minicomputers, technologisch èn economisch binnenkort tot de reële mogelijkheden lijkt te behoren.

2. Fysische Achtergronden

(Hoofdstukken 2 t/m 7)

In het navolgende wordt het schatten van evoked potentials eenvoudigheidshalve opgevat als het meer algemene probleem van het schatten van een onbekend *deterministisch* signaal, op basis van een gegeven *ensemble*, waarvan de elementen bestaan uit de invariante signaalcomponent en additieve , normaal verdeelde ruis. Gewoonlijk wordt het signaal geschat door middeling van het ensemble. De vraag is echter of er wellicht betere methoden (in de zin van de kleinste gemiddelde kwadratische fout) dan ensemble middeling bestaan.

Aan het eind van de zestiger jaren werd op zuiver heuristische gronden de "*a posteriori* 'Wiener' filtering" geintroduceerd. In deze methode wordt de na middeling verkregen signaalvorm bewerkt door een "optimaal" tijd-*in*variant filter dat, volgens de Wiener filtertheorie, wordt berekend uit de vermogensspectra van signaal en ruis. Anders dan in Wiener's oorspronkelijke theorie zijn deze spectra echter niet *(a priori)* bekend, maar moeten worden *geschat* uit het gemeten ensemble.

Hoofdstuk 3 beschrijft een aantal verschillende methoden die hiervoor kunnen worden toegepast. Een conceptueel eenvoudige, maar theoretisch minder optimale methode is om naast het normale gemiddelde een "alternerend" gemiddelde te berekenen. Hierbij worden opeenvolgende ensemble-elementen afwisselend bij dit gemiddelde opgeteld en ervan afgetrokken. Indien het signaal invariant is zal dit, bij een *even* aantal middelingen, wegvallen, waardoor een schatting van de (alternerend gemiddelde) ruis en zijn vermogensspectrum kan worden verkregen. Mét het spectrum van het normale ensemble gemiddelde is het dan mogelijk om ook het spectrum van het onderliggende signaal te schatten.

Het blijkt dat de berekening van het "Wiener" filter op basis van deze spectra aanleiding geeft tot verschillende problemen. Omdat de spectra van signaal en ruis *schattingen* zijn, met een inherente variabiliteit, vertoont ook het berekende filter een zekere variabiliteit; zelfs kan dit niet-fysische waarden aanemen. *Hoofdstuk 2* bevat een analyse van dit probleem, waarin onder meer wordt geconcludeerd dat spectrale "smoothing" een belangrijke rol speelt bij het verkrijgen van betrouwbare resultaten. Deze conclusie wordt vervolgens ondersteund door een aantal simulaties. Voorts wordt ingegaan op een aantal controversiële aspecten van *a posteriori* "Wiener" filtering; een discussie die wordt voortgezet in *hoofdstuk 4*. In dit hoofdstuk wordt tevens uiteengezet dat deze methode, vanwege het tijd-*in*variante karakter, minder geschikt is voor toepassing op de dynamische, transiente signaalvorm van evoked potentials in het algemeen.

Als voorbereiding op een gegeneraliseerde tijd-*variante* methode, beschrijft *hoofdstuk 5* een methodologische studie betreffende simultane frequentie-tijd *(spectro-temporele)* representaties. In het bijzonder worden in dit hoofdstuk verschillende methoden van tijd-variante spectrale analyse toegelicht en de voor evoked potentials meest geschikte keuze gemotiveerd.

Op basis van deze studie wordt in *hoofdstuk 6* de *(a posteriori)* tijd-variante filtermethode geintroduceerd en in *hoofdstuk 7* nader uitgewerkt. Hoofdstuk 7 beschrijft tevens een simulatie-studie waarin, voor een specifiek voorbeeld, een kwantitatieve vergelijking wordt gemaakt met normale middeling en *a posteriori* "Wiener" filtering. In dit voorbeeld blijkt tijd-variante filtering ten opzichte van ensemble middeling een verkleining van de gemiddelde kwadratische fout met een factor 1.5 à 5 (afhankelijk van de gekozen initiele signaal-ruis verhouding) op te leveren.

Een belangrijk deel van hoofdstuk 6 gaat in op de fundamentele discussie die in de literatuur ontstond naar aanleiding van publicatie van de hoofdstukken 2 en 3. Deze discussie betrof de vraag of het theoretisch wel mogelijk is om tot betere schattingen dan het normale ensemble gemiddelde te komen. Voor de beantwoording van deze vraag wordt in hoofdstuk 6 verwezen naar de theoretisch-statistische literatuur, waar dezelfde kwestie in de vijftiger jaren al tot ferme discussies aanleiding blijkt te hebben gegeven. Theoretisch kan worden aangetoond, dat er inderdaad betere schattingsmethoden dan middeling (de zgn. "empirische Bayes" schatters) bestaan, hoewel een algemene theorie nog onvoldoende uitgewerkt lijkt en in de praktijk de oplossingen nogal probleemgebonden zijn.

Het blijkt dat het *a posteriori* tijd-variante filter, evenals het *a posteriori* "Wiener" filter, een nauwe verwantschap vertoont met genoemde statistische schatters. Deze bevinding heeft veel bijgedragen tot het inzicht in de grondslagen van deze *a posteriori* filter methoden. Bovendien biedt deze verwantschap tal van aanknopingspunten met reeds langer bestaande *classificatie* (i.t.t. *schattings*-) methoden voor evoked potentials, waaronder de discriminant- en principal component-analyse. Studies waarin het bestaan van een *standaard normale evoked potential* wordt aangetoond werpen voorts een nieuw licht op het mogelijk gebruik van *a priori* informatie teneinde de schatting van gemiddelde evoked potentials per individu te verbeteren. Het lijkt mogelijk de in dit proefschrift beschreven filtertechnieken en al langer bestaande statistische methoden te integreren in een fundamenteel onderbouwde, krachtiger schattingstheorie. De uitwerking hiervan vereist een multidisciplinaire aanpak die de mogelijkheden én doelstellingen van een klinisch neurofysiologische afdeling ver te boven gaat, maar niettemin alleszins de moeite waard lijkt.

Curriculum vitae

Hans de Weerd werd geboren op 10 augustus 1947 te Den Haag. De lagere en middelbare school doorliep hij op Curacao (Nederlandse Antillen), waarheen het gezin in 1953 emigreerde. Na het behalen van het h.b.s.-b diploma aan het Radulphus College te Willemstad vertrok hij in 1964 naar Nederland voor een studie aan de afdeling Electrotechniek van de Technische Hogeschool te Eindhoven. Het kandidaatsexamen werd afgelegd in juli 1968. Hierna was hij gedurende 1 jaar parttime studentassistent bij prof.dr.ir. P. Eykhoff. De doctoraal opdracht werd verricht in het laboratorium voor Medische Fysica en Biofysica van de Universiteit te Nijmegen (hoofd prof.dr. A.J.H. Vendrik) onder leiding van ir. G.J.H. Uyen en betrof de detectie en analyse van hoogfrequente componenten in het electrocardiogram. Het ingenieursexamen legde hij cum laude af in januari 1971.

Van 1971 tot 1974 was hij vervolgens als wetenschappelijk medewerker verbonden aan bovengenoemd laboratorium met als deeltaken enerzijds het voortzetten van het tijdens de doctoraalfase verrichte onderzoek en anderzijds het opzetten van medisch-fysische ondersteuning ten behoeve van de afdeling Klinische Neurofysiologie (hoofd prof.dr. S.L.H. Notermans) van het Instituut voor Neurologie te Nijmegen. In april 1974 trad hij volledig in dienst van laatstgenoemde afdeling, waar hij sindsdien als werkgroepleider verantwoordelijk is voor het onderzoek van de afdeling op het gebied van kwantitatieve EEG en EMG analyse, evoked potential methodologie en electrofysiologische modelvorming, dit laatste in samenwerking met het laboratorium voor Medische Fysica en Biofysica.

STELLINGEN

I

Voor een optimale benutting van de mogelijkheden van "evoked potential" onderzoek voor de klinisch neurofysiologische diagnostiek, alsmede uit het oogpunt van dienstverlening aan de neurologische patient, is het van belang om naast de somatosensorische- ook de visuele- en auditieve modaliteit binnen één en dezelfde afdeling te kunnen onderzoeken.

II

Voor het schatten van een zich herhalend deterministisch signaal temidden van ruis bestaan betere methoden dan ensemble middeling.

C. Stein: Inadmissibility of the usual estimator for the mean of a multivariate normal distribution. Proc. Third Berkeley Symp. Math. Stat. and Prob. Vol. 1, University of California Press, 197-206, 1955.

Dit proefschrift, hoofdstuk 6.

III

Het feit dat spectrale computeranalyse van het electroencephalogram in de kliniek slechts in ongeveer 10% van de gevallen bijdraagt tot een verbeterde diagnostiek [1)] maakt, mede gelet op de noodzakelijke personele en materiële investeringen, een kritische bezinning aangaande de wenselijkheid van toepassing op grote schaal noodzakelijk.

[1)] E.J. Colon, J.P.C. de Weerd, N. v.d. Veer, J. Kap: Die normalen Provokationen im Leistungsspektrum. Eine klinisch-elektroenzephalographische Untersuchung. Z.EEG-EMG, 10, 132-136, 1979.

IV

In publikaties, die van de eigen afdeling niet uitgezonderd, betreffende de refractaire periode van perifere zenuwvezels, is tot nu toe aan de invloed van een der belangrijkste parameters, t.w. de toegepaste stimulus intensiteit, ten onrechte onvoldoende aandacht geschonken.

D.F. Stegeman: Compound nerve action potentials. An electrophysiological model study of human peripheral nerves *in situ*. Proefschrift, Nijmegen 1981.

V

Uit studies betreffende de electrische volumegeleiding in het menselijk lichaam valt te begrijpen waarom aan de amplitude van electrofysiologische signalen veel minder betekenis kan worden gehecht dan aan het moment van optreden van deze signalen.

VI

De tot nu toe bekende kwantitatieve analysemethoden voor het electromyografische interferentiepatroon leveren geen significante bijdrage tot een verbeterde diagnostiek.

VII

Methoden waarbij de waarde van wetenschappelijke publikaties voornamelijk wordt geschat op basis van de "Citation Index" zijn onzuiver.

VIII

Het bezuinigen op een van de meest essentiële aspekten van een promotieonderzoek, namelijk een representatieve verslaggeving, werkt demotiverend en is inefficiënt in het licht van de totale kosten van dergelijk onderzoek.

IX

Ondoordachte, op energiebesparing gerichte maatregelen, zoals de installatie van tijdschakelaars voor de verlichting in toiletten en trappenhuizen, wekken irritatie en hebben daarom op de lange termijn een averechtse uitwerking.

Juni 1981 J.P.C.M. de Weerd.